Electromagnetic induction. 
Faraday’s Law. Lentz Law. 
Maxwell’s Equations
Induction

• An *induced current* is produced by a *changing magnetic field*

• There is an induced emf associated with the induced current

• *A current can be produced without a battery present in the circuit*

• Faraday’s law of induction describes the induced emf
Current induced by a changing magnetic field

- A loop of wire is connected to a sensitive ammeter.
- When a magnet is moved toward the loop, the ammeter deflects.
  - The direction was arbitrarily chosen to be negative.
Current induced by a changing magnetic field (2)

• When the magnet is held stationary, there is no deflection of the ammeter
• Therefore, there is no induced current
  • Even though the magnet is in the loop
Current induced by a changing magnetic field (3)

- The magnet is moved away from the loop
- The ammeter deflects in the opposite direction
Current induced by a changing magnetic field (summary)

• The ammeter deflects when the magnet is moving toward or away from the loop
• The ammeter also deflects when the loop is moved toward or away from the magnet
• **Therefore, the loop detects that the magnet is moving relative to it**
  • We relate this detection to a change in the magnetic field
  • This is the induced current that is produced by an induced emf
Faraday’s experiment - setup

- A primary coil is connected to a switch and a battery
- The wire is wrapped around an iron ring
- A secondary coil is also wrapped around the iron ring
- There is no battery present in the secondary coil
- The secondary coil is not directly connected to the primary coil
Faraday experiment – findings

• At the instant the switch is closed, the ammeter changes from zero in one direction and then returns to zero
• When the switch is opened, the ammeter changes in the opposite direction and then returns to zero
• The ammeter reads zero when there is a steady current or when there is no current in the primary circuit
Faraday experiment – conclusions

• An electric current can be induced in a loop by a changing magnetic field
  • This would be the current in the secondary circuit of this experimental set-up

• The induced current exists only while the magnetic field through the loop is changing

• This is generally expressed as: *an induced emf is produced in the loop by the changing magnetic field*
  • *The actual existence of the magnetic flux is not sufficient to produce the induced emf, the flux must be changing*
Faraday’s law – statement

• Faraday’s law of induction states that “the emf induced in a circuit is directly proportional to the time rate of change of the magnetic flux through the circuit”

• Mathematically,

\[ \mathcal{E} = -\frac{d\Phi_B}{dt} \]
Faraday’s law (cont.)

- Remember $\Phi_B$ is the magnetic flux through the circuit and is found by
  $$\Phi_B = \int \mathbf{B} \cdot d\mathbf{A}$$

- If the circuit consists of $N$ loops, all of the same area, and if $\Phi_B$ is the flux through one loop, an *emf is induced in every loop* and Faraday’s law becomes
  $$\varepsilon = -N \frac{d\Phi_B}{dt}$$
Faraday’s law – example

• Assume a loop enclosing an area $A$ lies in a uniform magnetic field
• The magnetic flux through the loop is $\Phi_B = BA \cos \theta$
• The induced emf is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(BA \cos \theta)$$
Ways of inducing EMF

- The magnitude of $\vec{B}$ can change with time
- The area enclosed by the loop can change with time
- The angle $\theta$ between $\vec{B}$ and the normal to the loop can change with time
- Any combination of the above can occur
Induced EMF and electric field

• An electric field is created in the conductor as a result of the changing magnetic flux
• Even in the absence of a conducting loop, a changing magnetic field will generate an electric field in empty space!
• This induced electric field is nonconservative
  • Unlike the electric field produced by stationary charges
Induced EMF and electric field (2)

- The emf for any closed path can be expressed as the line integral of $\mathbf{E} \cdot d\mathbf{l}$ over the path (recall work done by electric field)

- *Faraday’s law can be written in a general form:*

$$
\mathcal{E} = \int \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}
$$
Induced EMF and electric field (3)

- The induced electric field is a nonconservative field that is generated by a changing magnetic field.
- The field cannot be an electrostatic field because if the field were electrostatic, and hence conservative, the line integral of $\mathbf{E} \cdot d\mathbf{s}$ would be zero and it isn’t.
Example: electric field induced by a changing magnetic field in a solenoid

- Solenoid of radius $R$ has $n$ turns per unit length and carries a time-varying current $I = I_{\text{max}} \cos \omega t$. Determine the magnitude of the induced electric field outside the solenoid at a distance $r > R$ from its long central axis.
Electric field induced by solenoid (2)

- Because current varies in time the magnetic field inside solenoid changes. This would generate electric field in a surrounding space.

- Faraday’s law:

\[ \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \]

- Construct a circular path of radius \( r \) around solenoid

- Magnetic flux through the area enclosed by this path

\[ \Phi_B = B \pi R^2 \]

Magnetic field is nonzero only inside solenoid
Electric field induced by solenoid (3)

- Since current is nonstationary magnetic field inside solenoid is also a function of time
  \[ B = \mu_0 n I = \mu_0 n I_{\text{max}} \cos(\omega t) \]

- Substituting the expression for magnetic field, the magnetic flux becomes
  \[ \Phi_B = \pi R^2 \mu_0 n I_{\text{max}} \cos(\omega t) \]

- From the symmetry, at every time instance \( E \) is constant along the path and aligned with the path element \( ds \)
  \[ \oint E \cdot ds = 2\pi r E = -\frac{d\Phi_B}{dt} = \pi R^2 \mu_0 n \omega I_{\text{max}} \sin(\omega t) \]

- Electric field outside solenoid
  \[
  E = \frac{R^2 \mu_0 n \omega I_{\text{max}} \sin(\omega t)}{2r}
  \]
Lenz’s law

- Faraday’s law indicates that the induced emf and the change in flux have opposite algebraic signs.
- This has a *physical interpretation* that has come to be known as Lenz’s law.
- Developed by German physicist Heinrich Lenz.
Lenz’s law - formulation

- **Lenz’s law**: the induced current in a loop is in the direction that creates a magnetic field that opposes the change in magnetic flux through the area enclosed by the loop.

- The induced current tends to keep the original magnetic flux through the circuit from changing.
Terminology

- Use **emf** and **current** when they are caused by batteries or other sources
- Use **induced emf** and **induced current** when they are caused by changing magnetic fields
- When dealing with problems in electromagnetism, it is important to distinguish between the two situations
Lenz’s law - example

- The conducting bar slides on the two fixed conducting rails
- The magnetic flux due to the external magnetic field through the enclosed area increases with time
- *The induced current must produce a magnetic field out of the page*
  - The induced current must be counterclockwise
- If the bar moves in the opposite direction, the direction of the induced current will also be reversed
Example: magnetic force acting on a sliding bar

- The conducting bar moves on two frictionless, parallel rails in the presence of a uniform magnetic field directed into the page. The bar has mass $m$, and its length is $l$. The bar is given an initial velocity $v_i$ to the right and is released at $t=0$.

- Using Newton’s laws, find the velocity of the bar as a function of time.
Magnetic force acting on a sliding bar

• Applying second Newton’s law:

\[ F_x = ma = m \frac{dv}{dt} = -IlB \]

• Current generated in the loop:

\[ I = Blv/R \]

• After substitution:

\[ m \frac{dv}{dt} = -\frac{B^2l^2}{R} v \]

• Integrating left and right hand sides:

\[ m \int_{v_i}^{v} \frac{dv}{v} = -\frac{B^2l^2}{R} \int_{0}^{t} dt \quad \Rightarrow \quad m \ln \left( \frac{v}{v_i} \right) = -\frac{B^2l^2}{R} t \]
Magnetic force acting on a sliding bar (2)

• The solution is:

\[ v = v_i e^{-t/\tau} \]

where

\[ \tau = \frac{mR}{B^2l^2} \]

• The velocity decays with time
Self inductance

- When the switch is closed, the current does not immediately reach its maximum value.
- Faraday’s law can be used to describe the effect.
- *As the current increases with time, the magnetic flux through the circuit loop due to this current also increases with time.*
- This increasing flux creates an *induced emf* in the circuit.
Self inductance - equation

• An induced emf is always proportional to the time rate of change of the current
  • The emf is proportional to the flux, which is proportional to the field and the field is proportional to the current

$$\varepsilon_L = -\frac{d\Phi_B}{dt} = -\frac{\Phi_B}{I} \frac{dI}{dt}$$

$$\varepsilon_L = -L \frac{dI}{dt}$$

• $L$ is a constant of proportionality called the inductance of the coil and it depends on the geometry of the coil and other physical characteristics
Inductance of a coil

- The inductance is a measure of the opposition to a change in current
- The SI unit of inductance is the *henry* (H)
  - Named for Joseph Henry

\[ 1 \text{H} = 1 \frac{\text{V} \cdot \text{s}}{\text{A}} \]
Inductance of a solenoid

• Assume a uniformly wound solenoid having \( N \) turns and length \( l \)

• The flux through each turn of area \( A \) is

\[
\Phi_B = BA = \mu_0 nIA = \mu_0 \frac{N}{l} IA
\]

• The inductance is

\[
L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 A}{l} = \mu_0 n^2 V
\]

• This shows that \( L \) depends on the geometry of the object
  • \( L \) is proportional to the volume \( V \) of a solenoid
Effect of an inductor in a circuit

- The inductance results in a back emf.
- Therefore, *the inductor in a circuit opposes changes in current in that circuit*
  - The inductor attempts to keep the current the same way it was before the change occurred.
  - The inductor can cause the circuit to be “sluggish” as it reacts to changes in the voltage.
RL circuit – analysis

- An RL circuit contains an inductor and a resistor
- Assume $S_2$ is connected to $a$
- When switch $S_1$ is closed (at time $t = 0$), the current begins to increase
- At the same time, a back emf is induced in the inductor that opposes the original increasing current
RL circuit – analysis (2)

• The voltage balance across the loop is

\[ \varepsilon - IR - L \frac{dI}{dt} = 0 \]

• Solution for current

\[ I = \frac{\varepsilon}{R} \left( 1 - e^{-\frac{Rt}{L}} \right) \]
• The inductor affects the current exponentially
• *The current does not instantly increase to its final equilibrium value*
• If there is no inductor, the exponential term goes to zero and the current would instantaneously reach its maximum value as expected
**RL circuit – current-time graph**

- The equilibrium value of the current is $\varepsilon/R$ and is reached as $t$ approaches infinity.
- The current initially increases very rapidly.
- The current then gradually approaches the equilibrium value.
RL circuit without a battery

• Now set $S_2$ to position $b$
• The circuit now contains just the right hand loop
• The battery has been eliminated
• The expression for the current becomes

$$I = I_i e^{-\frac{Rt}{L}} = \frac{\mathcal{E}}{R} e^{-\frac{Rt}{L}}$$
Energy in a magnetic field

- In a circuit with an inductor, the battery must supply more energy than in a circuit without an inductor.
- Part of the energy supplied by the battery appears as internal energy in the resistor.
- The remaining energy is stored in the magnetic field of the inductor.
Energy in a magnetic field (2)

- Looking at this energy (in terms of rate)

\[ I\varepsilon - I^2 R - LI \frac{dI}{dt} = 0 \]

- \( I\varepsilon \) is the rate at which energy is being supplied by the battery
- \( I^2 R \) is the rate at which the energy is being delivered to the resistor
- Therefore, \( LI \frac{dI}{dt} \) must be the rate at which the energy is being stored in the magnetic field
Energy in a magnetic field (3)

- Let $U$ denote the energy stored in the inductor at any time.
- The rate at which the energy is stored is

$$\frac{dU}{dt} = LI \frac{dI}{dt}$$

- To find the total energy, integrate and

$$U = L \int_{0}^{I} I dI = \frac{1}{2} LI^2$$
Energy in a magnetic field (4)

- Given $U = \frac{1}{2} LI^2$ and assume (for simplicity) a solenoid with $L = \mu_0 n^2 V$

$$U = \frac{1}{2} LI^2 = \frac{1}{2} \mu_0 n^2 \left( \frac{B}{\mu_0 n} \right)^2 V = \frac{B^2}{2\mu_0} V$$

- Since $V$ is the volume of the solenoid, the magnetic energy density, $u_B$ is

$$u_B = \frac{B^2}{2\mu_0}$$

- This applies to any region in which a magnetic field exists (not just the solenoid)
Energy storage – summary

• A resistor, inductor and capacitor all store energy through different mechanisms
  • Charged capacitor
    • Stores energy as electric potential energy
  • Inductor
    • When it carries a current, stores energy as magnetic potential energy
  • Resistor
    • Energy delivered is transformed into internal energy
James Clerk Maxwell

- 1831 – 1879
- Scottish physicist
- Provided a mathematical theory that showed a close relationship between all electric and magnetic phenomena
- His equations predict the existence of electromagnetic waves that propagate through space
- Also developed and explained
  - Kinetic theory of gases
  - Nature of Saturn’s rings
  - Color vision
Discontinuity of current

• Path $P$ enclosing surface $S_1$
  \[ \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I \]

• Path $P$ enclosing surface $S_2$
  \[ \oint \mathbf{B} \cdot d\mathbf{s} = 0 \]

• Contradiction arises from discontinuity of current!
Modifications to Ampère’s Law

• Ampère’s Law is used to analyze magnetic fields created by currents:

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 I \]

• But this form is valid only if any electric fields present are constant in time

• Maxwell modified the equation to include time-varying electric fields
Displacement current

- Current is the rate of change of charge
  
  \[ I = \frac{dq}{dt} \]

- Electric flux through the surface \( S_2 \)
  
  \[ \Phi_E = \frac{q}{\epsilon_0} \]

- Displacement current
  
  \[ I_d = \frac{dq}{dt} = \epsilon_0 \frac{d\Phi_E}{dt} \]
Modifications to Ampère’s Law (2)

- The additional term included in the Ampere’s law is called the displacement current, $I_d$

\[ I_d = \epsilon_0 \frac{d\Phi_E}{dt} \]

- Now sometimes called Ampère-Maxwell Law
- $\Phi_E$ is the electric flux through the surface enclosed by path
- This showed that magnetic fields are produced both by conduction currents and by time-varying electric fields

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \]
Maxwell’s equations – Gauss’s law

• The total electric flux through any closed surface equals the net charge inside that surface divided by $\varepsilon_0$

$$\iiint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0}$$

• This relates an electric field to the charge distribution that creates it
Maxwell’s equations – Gauss’s law in magnetism

• The net magnetic flux through a closed surface is zero

\[ \oint \mathbf{B} \cdot d\mathbf{A} = 0 \]

• The number of magnetic field lines that enter a closed volume must equal the number that leave that volume

• If this wasn’t true, there would be magnetic monopoles found in nature
  • There haven’t been any found
Maxwell’s equations – Faraday’s law of induction

- Describes the creation of an electric field by a time-varying magnetic field
- The emf, which is the line integral of the electric field around any closed path, equals the rate of change of the magnetic flux through any surface bounded by that path

\[ \int \mathbf{E} \cdot d\mathbf{s} = - \frac{d\Phi_B}{dt} \]

- One consequence is the current induced in a conducting loop placed in a time-varying magnetic field
Maxwell’s equations – Ampere’s-Maxwell law

- Describes the creation of a magnetic field by a changing electric field and by electric current
- The line integral of the magnetic field around any closed path is the sum of $\mu_0$ times the net current through that path and $\epsilon_0\mu_0$ times the rate of change of electric flux through any surface bounded by that path

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$$
Maxwell’s equations - summary

- Maxwell’s equations combine laws of electricity and magnetism:
  - Gauss’s law
    \[ \iiint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0} \]
  - Gauss’s law in magnetism
    \[ \iiint \vec{B} \cdot d\vec{A} = 0 \]
  - Faraday’s law
    \[ \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \]
  - Ampere’s-Maxwell law
    \[ \oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \]