Sources of Magnetic Field
Biot-Savart Law – introduction

• Biot and Savart conducted experiments on the force exerted by an electric current on a nearby magnet

• They arrived at a mathematical expression that gives the magnetic field at some point in space due to a current
Biot-Savart Law – observations

• The vector $d\vec{B}$ is perpendicular to both $d\vec{I}$ and to the unit vector $\hat{r}$ directed from $d\vec{I}$ toward $P$.
• The magnitude of $d\vec{B}$ is inversely proportional to $r^2$, where $r$ is the distance from $d\vec{I}$ to $P$.
• The magnitude $dB$ is proportional to the current $I$ and to the magnitude $dl$.
• $dB$ is proportional to the $\sin \theta$ where $\theta$ is angle between $d\vec{I}$ and $\hat{r}$. 
Biot-Savart Law – equation

• The observations are summarized in the mathematical equation called the

\textit{Biot-Savart law}:

\[
d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}
\]

• The magnetic field described by the law is the field \textit{due} to the current-carrying conductor

• Don’t confuse this field with a field \textit{external} to the conductor

• The constant \(\mu_0\) is called the \textit{permeability of free space}

\[
\mu_0 = 4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}
\]
Total magnetic field

- \( d\mathbf{B} \) is the field created by the current in the length segment \( dl \)
- To find the total field, sum up the contributions from all the current elements

\[
\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \hat{r}}{r^2}
\]

- The integral is over the entire current distribution
- The integral contains the direction
Direction of magnetic field

**Right-hand rule for the magnetic field due to a current element:** Point the thumb of your right hand in the direction of the current. Your fingers now curl around the current element in the direction of the magnetic field lines.

For these field points, \( \vec{r} \) and \( d\vec{l} \) both lie in the beige plane, and \( d\vec{B} \) is perpendicular to this plane.

(b) View along the axis of the current element

For these field points, \( \vec{r} \) and \( d\vec{l} \) both lie in the gold plane, and \( d\vec{B} \) is perpendicular to this plane.
Magnetic field due to point charge

- From Biot-Savart law
- Current $I = qv_d nA$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{l} \times \hat{r}}{r^2}$$

$$q n A d l = q N = Q$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{Q\vec{v} \times \hat{r}}{r^2}$$
$B$ compared to $E$

**Distance**
- The magnitude of the magnetic field varies as the inverse square of the distance from the source.
- The electric field due to a point charge also varies as the inverse square of the distance from the charge.

**Direction**
- The electric field created by a point charge is *radial* in direction.
- The magnetic field created by a current element is *perpendicular* to both the length element $d\vec{s}$ and the unit vector $\hat{r}$. 
The magnetic field $B$ compared to the electric field $E$

- **Source**
  - An electric field is established by an isolated electric charge.
  - The current element that produces a magnetic field *must* be part of an extended current distribution.
    - Therefore you must integrate over the entire current distribution.
**B** for a long straight conductor

- The thin, straight wire is carrying a constant current $I$. Find the magnetic field created by the wire at point $P$.

Solution:
- Apply B-S law:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{s} \times \hat{r}}{r^2}$$

- The vector product

$$d\vec{s} \times \hat{r} = \left[ dx \sin \left( \frac{\pi}{2} - \theta \right) \right] \hat{k} = [dx \cos \theta] \hat{k}$$
\[ \mathbf{B} \text{ for a long straight conductor} \]

- Field created by current element \( ds \)

\[
d\mathbf{B} = (dB)\hat{k} = \frac{\mu_0}{4\pi} \frac{I d\mathbf{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{dx \cos \theta}{r^2} \hat{k}
\]

- Distance \( r \) from \( ds \) to \( P \)

\[
r = \frac{a}{\cos \theta}
\]

\[
x = -a \tan \theta \quad \quad dx = -\frac{a d\theta}{\cos^2 \theta}
\]

- Magnitude of magnetic field as a function of \( \theta \)

\[
dB = \frac{\mu_0 I}{4\pi} \frac{dx \cos \theta}{r^2} = -\frac{\mu_0 I}{4\pi a} \cos \theta \ d\theta
\]
\( B \) for a long straight conductor

- Total field due to the wire
  \[
  B = -\frac{\mu_0 I}{4\pi a} \int_{\theta_1}^{\theta_2} \cos \theta \, d\theta = \frac{\mu_0 I}{4\pi a} (\sin \theta_1 - \sin \theta_2)
  \]

  \[\vec{B} = \frac{\mu_0 I}{4\pi a} (\sin \theta_1 - \sin \theta_2) \hat{k}\]

- If the conductor is an \textit{infinitely long, straight wire}, \( \theta_1 = \pi/2 \) and \( \theta_2 = -\pi/2 \). The field becomes
  \[
  B = \frac{\mu_0 I}{2\pi a}
  \]
Field on the axis of current loop

- The loop has a radius of \( R \) and carries a steady current of \( I \). Find the field at point \( P \).

- Biot-Savart law:
  \[
  d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I d\mathbf{s} \times \hat{r}}{r^2}
  \]

- Vector \( d\mathbf{s} \) is perpendicular to \( \hat{r} \), thus \( |d\mathbf{s} \times \hat{r}| = ds \)
Field on the axis of current loop

• Magnitude of the magnetic field created by current element $ds$

$$dB = \frac{\mu_0 I}{4\pi} \frac{ds}{r^2} = \mu_0 I \frac{ds}{4\pi} \frac{ds}{a^2 + x^2}$$

• The $x$-component of magnetic field

$$dB_x = \frac{\mu_0 I}{4\pi} \frac{ds}{a^2 + x^2} \cos \theta$$
Field on the axis of current loop

- The $x$-component of total magnetic field

$$B_x = \oint dB_x = \frac{\mu_0 I}{4\pi} \frac{\cos \theta}{a^2 + x^2} \int ds = \frac{\mu_0 I}{4\pi} \frac{\cos \theta}{a^2 + x^2} 2\pi a$$

- Based on the symmetry, $y$-component of the total field must be zero

- Total magnetic field at $P$

$$\vec{B} = \frac{\mu_0 I a^2}{2(a^2 + x^2)^{3/2}} \hat{i}$$

$$\cos \theta = \frac{a}{\sqrt{a^2 + x^2}}$$
Magnetic field lines for a loop

- Figure (a) shows the magnetic field lines surrounding a current loop
- Figure (b) shows the field lines in the iron filings
- Figure (c) compares the field lines to that of a bar magnet
Magnetic force between two parallel conductors

- Two long parallel wires each carry a steady current
- The field $\vec{B}_2$ due to the current in wire 2 exerts a force on wire 1 of

$$F_1 = I_1 l B_2$$
Magnetic force between two parallel conductors (2)

• Substituting the equation for $\vec{B}_2$ gives

$$F_1 = \frac{\mu_0 I_1 I_2}{2\pi a} l$$

• Parallel conductors carrying currents in the same direction attract each other

• Parallel conductors carrying current in opposite directions repel each other
Ampere’s Law
Magnetic field of a wire (3)

- The circular magnetic field around the wire is shown by the iron filings

\[ B = \frac{\mu_0 I}{2\pi r} \]
Ampère’s law

• Ampere’s law states that the line integral of \( \mathbf{B} \cdot d\mathbf{l} \) around any closed path equals \( \mu_0 I \) where \( I \) is the total steady current passing through any surface bounded by the closed path (amperian loop):

\[
\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I
\]
Ampere’s Law - derivation

- Consider arbitrary path enclosing current $I$
- The scalar product:
  \[ \vec{B} \cdot d\vec{l} = Bdl \cos \phi \]
- From the plot: $dl \cos \phi = rd\theta$
- Magnetic field created by conductor with current $I$ at a distance $r$
  \[ B = \frac{\mu_0 I}{2\pi r} \]
- Line integral along the path
  \[ \oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi r} (rd\theta) = \frac{\mu_0 I}{2\pi} \oint d\theta = \mu_0 I \]
Ampere’s Law - derivation

- If the path does not enclose the wire, then the net change in $\theta$ during the trip around the integration path is zero.

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint d\theta = 0 \]
Ampere’s Law – derivation

- If there is more than one wire, the magnetic field is equal to the vector sum of field created by each wire

\[ \oint (\vec{B}_1 - \vec{B}_2 + \vec{B}_3) \cdot d\vec{l} = \mu_0 (I_1 - I_2 + I_3) = \mu_0 I_{\text{enc}} \]

\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} \]
Ampère’s law – application

• Ampere’s law describes the creation of magnetic fields by all continuous current configurations
  • Most useful if the current configuration has a high degree of symmetry

• Put the thumb of your right hand in the direction of the current through the amperian loop and your fingers curl in the direction you should integrate around the loop
Field due to a long straight wire – from Ampere’s law

- Calculate the magnetic field at a distance $r$ from the center of a wire carrying a steady current $I$
- The current is uniformly distributed through the cross section of the wire
Field due to a long straight wire – from Ampere’s law (cont.)

- **Outside of the wire, \( r > R \)**

\[
\oint \vec{B} \cdot d\vec{l} = B2\pi r = \mu_0 I \quad \implies \quad B = \frac{\mu_0 I}{2\pi r}
\]

- **Inside the wire, \( r < R \)**

\[
I_{\text{enc}} = \sigma A = \sigma \pi r^2
\]

- \( \sigma \) – current per unit area

\[
\sigma = \frac{I}{\pi R^2}
\]

\[
B = \frac{\mu_0 I_{\text{enc}}}{2\pi r} = \frac{\mu_0 I}{2\pi R^2} r
\]
Field due to a long straight wire – from Ampere’s law (summary)

- The field is proportional to $r$ inside the wire.
- The field varies as $1/r$ outside the wire.
- Both equations are equal at $r = R$. 

![Diagram showing field $B$ varying with $r$ inside and outside the wire with $B \propto r$ inside and $B \propto 1/r$ outside.](image)
A **solenoid** is a long wire wound in the form of a helix.

A reasonably uniform magnetic field can be produced in the space surrounded by the turns of the wire:

- The interior of the solenoid
Magnetic field of a solenoid (2)

- The field lines in the interior of solenoid are
  - nearly parallel to each other
  - uniformly distributed
  - close together
- This indicates the field is strong and almost uniform
Magnetic field of a solenoid (3)

- *The field distribution is similar to that of a bar magnet*

- As the length of the solenoid increases
  - the interior field becomes more uniform
  - the exterior field becomes weaker

- An ideal solenoid is approached when:
  - the turns are closely spaced
  - the length is much greater than the radius of the turns
Magnetic field of a solenoid (4)

• Current passing through the loop 1 is $I$. The magnetic field outside the solenoid is small.

• The outside field is in the perpendicular plane.
Magnetic field of a solenoid (5)

- Consider a rectangle with side \( l \) parallel to the interior field and side \( w \) perpendicular to the field (loop 2)
- Side 2, 3, and 4 are perpendicular to the magnetic field

\[
\vec{B} \cdot d\vec{l} = 0
\]
Magnetic field of a solenoid (6)

- Only side of length \( l \) inside the solenoid contributes to the field
  - This is side 1 in the diagram

\[
\oint \mathbf{B} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{l} = Bl
\]

side 1

- The total current through the rectangular path equals the current through each turn multiplied by the number of turns

\[
\oint \mathbf{B} \cdot d\mathbf{l} = Bl = \mu_0 NI
\]
Magnetic field of a solenoid (7)

- Solving Ampere’s law for the magnetic field is
  \[ B = \frac{\mu_0 NI}{l} = \mu_0 nI \]
- \( n=N/l \) is the number of turns per unit length
- This is valid only at points near the center of a very long solenoid