Electric potential
Work done by electric field

- When a test charge is placed in an electric field, it experiences a force
  \[ \vec{F} = q_0 \vec{E} \]
- Work done by electric field while moving charge \( q_0 \) from A to B
  \[ \int_A^B \vec{F} \cdot d\vec{S} = q_0 \int_A^B \vec{E} \cdot d\vec{S} \]
  - \( d\vec{S} \) is an infinitesimal displacement vector
- Electric field is \textit{conservative} – work is independent of path
Electrical potential energy

- Work done by electric field
  
  \[ q_0 \vec{E} \cdot d\vec{S} \]

- As this work is done by the field, the potential energy of the \textit{charge-field system} is changed by
  
  \[ \Delta U = -q_0 \vec{E} \cdot d\vec{S} \]

- For a finite displacement of the charge from A to B,
  
  \[ \Delta U = U_B - U_A = -q_0 \int_A^B \vec{E} \cdot d\vec{S} \]

- Because the force is conservative, the line integral does not depend on the path taken by the charge
Electric potential

• The potential energy per unit charge, \( V = U/q_0 \), is the **electric potential**
  
  • *The potential is characteristic of the field only*
  
  • *The potential energy is characteristic of the charge-field system*

• The potential is independent of the value of \( q_0 \)

• The potential has a value at every point in an electric field

• As a charged particle moves in an electric field, it will experience a change in potential

\[
\Delta V = \frac{\Delta U}{q_0} = -\int_{A}^{B} \vec{E} \cdot d\vec{S}
\]
Electric potential (cont.)

• The *difference* in potential is the meaningful quantity
• We often take the value of the potential to be zero at some convenient point in the field
• Electric potential is a scalar characteristic of an electric field, independent of any charges that may be placed in the field
Work and electric potential

- Assume a charge moves in an electric field without any change in its kinetic energy.
- The work performed on the charge is

\[ W = \Delta U = q \Delta V \]
Units

- 1 V = 1 J/C
  - V is a volt
  - It takes one joule of work to move a 1 coulomb charge through a potential difference of 1 volt
- In addition, 1 N/C = 1 V/m
  - This indicates we can interpret the electric field as a measure of the rate of change with position of the electric potential
Electron-Volt

• Another unit of energy that is commonly used in atomic and nuclear physics is the electron-volt

• One **electron-volt** is defined as the energy a charge-field system gains or loses when a charge of magnitude \( e \) (an electron or a proton) is moved through a potential difference of 1 volt

  • \( 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J} \)
Potential difference in uniform field

• The equations for electric potential can be simplified if the electric field is uniform:

\[ V_B - V_A = \Delta V = - \int_A^B \mathbf{E} \cdot d\mathbf{S} = -E \int_A^B dS = -Ed \]

• The negative sign indicates that the electric potential at point \( B \) is lower than at point \( A \)

  • *Electric field lines always point in the direction of decreasing electric potential*
Energy and direction of electric field

- When the electric field is directed downward, point $B$ is at a lower potential than point $A$.
- When a *positive* test charge moves from $A$ to $B$, the charge-field system loses potential energy.
Energy conservation

- A system consisting of a positive charge and an electric field **loses** potential energy when the charge moves in the direction of the field
  - An electric field does work on a positive charge when the charge moves in the direction of the field
- A system consisting of a negative charge and an electric field **gains** potential energy when the charge moves in the direction of the field
  - In order for a negative charge to move in the direction of the field, an external agent must do positive work on the charge
- *The charged particle gains* kinetic energy *equal to the potential energy lost by the charge-field system and vice-versa*
Equipotentials

- Point $B$ is at a lower potential than point $A$
- Points $B$ and $C$ are at the same potential
  - All points in a plane perpendicular to a uniform electric field are at the same electric potential
- The name *equipotential surface* is given to any surface consisting of a continuous distribution of points having the same electric potential

$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{S}$$
Potential and point charges

- A positive point charge produces a field directed radially outward
- The potential difference between points A and B will be

\[
V_B - V_A = \int_A^B \mathbf{E} \cdot d\mathbf{S}
\]

\[
\mathbf{E} \cdot d\mathbf{S} = k_e \frac{q}{r^2} \hat{r} \cdot d\mathbf{S}
\]

\[
\hat{r} \cdot d\mathbf{S} = ds \cos \theta = dr
\]

\[
V_B - V_A = k_e q \int_A^B \frac{dr}{r^2} = k_e q \left[ \frac{1}{r_B} - \frac{1}{r_A} \right]
\]
Potential and point charges (cont.)

• The potential difference is independent of the path between points A and B
• It is customary to choose a reference potential of $V = 0$ at $r_A = \infty$
• Then the potential at some point $r$ is

$$V = \frac{k_e q}{r}$$

• Electric potential due to several point charges is the sum of the potentials due to each individual charge

$$V = k_e \sum_i \frac{q_i}{r_i}$$
Potential and point charges (final)

- Electric potentials of single positive point charge and dipole
Example: moving through a potential difference

- Consider charged particle with mass \( m=5 \mu g \) and charge \( q_0=2\text{nC} \) moving from point \( a \) to point \( b \) in the electric field created by two point charges. The initial velocity at point \( a \) is zero. Find the velocity at point \( b \).

- The energy is conserved

\[
K_a + U_a = K_b + U_b
\]

\[
K = \frac{mv^2}{2} \quad \quad U = q_0V
\]
Moving through a potential difference

\[ K_a = 0 \quad K_b = \frac{mv^2}{2} \quad U_a = q_0V_a \quad U_b = q_0V_b \]

\[ v = \sqrt{\frac{2q_0(V_a - V_b)}{m}} \]

Electric potential:

\[ V = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i} \]

\[ V_a = 9 \times 10^9 \times \left( \frac{3 \times 10^{-9}}{0.01} - \frac{3 \times 10^{-9}}{0.02} \right) = 1350V \]

\[ V_b = 9 \times 10^9 \times \left( \frac{3 \times 10^{-9}}{0.02} - \frac{3 \times 10^{-9}}{0.01} \right) = -1350V \]

\[ v = \sqrt{\frac{2 \times 2 \times 10^{-9} \times (1350+1350)}{5 \times 10^{-9}}} = 46 \text{ m/s} \]
Consider a small charge element $dq$

- Treat it as a point charge

- The potential at some point due to this charge element is

$$dV = k_e \frac{dq}{r}$$

- To find the total potential, you need to integrate to include the contributions from all the elements

$$V = k_e \int \frac{dq}{r}$$
Example: ring of charge

- Electric charge is distributed uniformly around a thin ring of radius $a$, with total charge $Q$. Find the potential at point $P$ at a distance $x$ as shown in the figure.

- Solution:
  - Separate a small segment. Potential created by this segment is
    \[ dV = k_e \frac{dq}{r} \]
  - The distance $r$ is the same for each segment on the ring
  - Total potential at point $P$
    \[ V = \int dV = k_e \int \frac{dq}{r} = \frac{k_e}{\sqrt{x^2 + a^2}} \int dq = \frac{k_e Q}{\sqrt{x^2 + a^2}} \]
Finding $E$ from $V$

• Assuming that field has only $x$-component $E_x$

• Potential difference between two points a distance $dS$ apart

$$dV = -\mathbf{E} \cdot d\mathbf{S} = -E_x dx$$

$$E_x = -\frac{dV}{dx}$$

• Similar statement would apply for $y$ and $z$ components

• Given $V(x, y, z)$ you can find $E_x$, $E_y$ and $E_z$ as partial derivatives

$$E_x = -\frac{\partial V}{\partial x} \quad E_y = -\frac{\partial V}{\partial y} \quad E_z = -\frac{\partial V}{\partial z}$$
Potential gradient

• Gradient of potential is a vector defined as

$$\vec{\nabla}V = \hat{i} \frac{\partial V}{\partial x} + \hat{j} \frac{\partial V}{\partial y} + \hat{k} \frac{\partial V}{\partial z}$$

• Gradient is directed toward increase of potential
• Electric field directed toward decrease of potential

$$\vec{E} = -\vec{\nabla}V$$
Example: ring of charge (cont.)

- Potential at point \( P \):
  \[
  V = \frac{k_e Q}{\sqrt{x^2 + a^2}}
  \]

- Electric field at point \( P \):
  \[
  \vec{E} = -\hat{i} \frac{\partial V}{\partial x} - \hat{j} \frac{\partial V}{\partial x} - \hat{k} \frac{\partial V}{\partial x}
  \]
  \[
  E_x = -\frac{\partial V}{\partial x} = k_e Q \left(\frac{x}{(x^2 + a^2)^{3/2}}\right)
  \]
  \[
  E_y = -\frac{\partial V}{\partial y} = 0
  \]

- Compare to Example 21.10
Potential due to charged conductor

- Consider two points on the surface of the charged conductor as shown
- \( \mathbf{E} \) is always perpendicular to the displacement \( d\mathbf{S} \)
- Therefore, \( \Delta V = -\mathbf{E} \cdot d\mathbf{S} = 0 \)
- Therefore, the potential difference between A and B is also zero
Potential due to charged conductor (cont.)

- $V$ is constant everywhere on the surface of a charged conductor in equilibrium
  - $\Delta V = 0$ between any two points on the surface
- *The surface of any charged conductor in electrostatic equilibrium is an equipotential surface*
- Because the electric field is zero inside the conductor, we conclude that the electric potential is constant everywhere inside the conductor and equal to the value at the surface
Electric field compared to potential

- The electric potential is a function of $r$
- The electric field is a function of $r^2$
- The effect of a charge on the space surrounding it:
  - The charge sets up a vector electric field which is related to the force
  - The charge sets up a scalar potential which is related to the energy
Cavity in a conductor

- Assume an irregularly shaped cavity is inside a conductor
- Assume no charges are inside the cavity
- For all paths between $A$ and $B$,

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{S} = 0$$

- Path is arbitrary thus the electric field inside the cavity must be zero
Potential energy of multiple charges

- Consider two charged particles
- Potential in point $P$ due to charge $q_2$
  \[ V_2 = \frac{k_e q_2}{r_{12}} \]
- Work *external agent* must do to bring $q_1$ from infinity to $P$ is $W = q_1 V_2$
- This work represents a transfer of energy *into* the system
- The potential energy of the system is
  \[ U = k_e \frac{q_1 q_2}{r_{12}} \]
Potential energy of multiple charges (cont.)

• If the two charges are the same sign, $U$ is positive and **work must be done to bring the charges together**

• If the two charges have opposite signs, $U$ is negative and **work is done to keep the charges apart**
Potential energy of multiple charges (final)

- If there are more than two charges, then find $U$ for each pair of charges and add them.
- For three charges:

$$U = k_e \left( \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

- The result is independent of the order of the charges.