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Pushforward transformation of gyrokinetic moments under electromagnetic fluctuations

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Pushforward transformation is one of the two important transformations in modern nonlinear gyrokinetic theory. In this work, a gyrokinetic system under electromagnetic fluctuations has been derived using a purely pushforward transformation, where the finite Larmor radius (FLR) effect is fully retained. From the perspective of polarization and magnetization, clear physical pictures of macroscopic equilibrium flow are presented, and the generation of macroscopic perturbed flow is discussed with the incorporation of the full FLR effect using the systematical analysis of the gyrocenter gyroradius and the decoupling of particle velocity. *Published by AIP Publishing.*
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I. INTRODUCTION

Gyrokinetic theory was developed as a generalization of guiding-center theory[1] to describe plasma processes over time scales that are longer than the gyromotion time scale in the presence of perturbed fluctuations. As an important tool in plasma physics research, traditional nonlinear gyrokinetic theory was first presented in the pioneering work of Frieman and Chen,[2] which was built upon linear gyrokinetic theory.[3–6] To construct a gyrokinetic theory that inherently holds energy conservation and Liouville's theorem, the Hamilton system[7] was proposed and introduced to investigate guiding-center dynamics,[8,9] where a Darboux transformation was used in noncanonical coordinates in phase space. Based on the Lie transform perturbation method for Hamiltonian systems, modern nonlinear gyrokinetic theory[10–14] was developed using perturbed gyrocenter Hamiltonian dynamics, which, in principal, can be easily expanded to any order.

Gyrokinetic theory has been extensively used as a powerful analytical tool in both laboratory and space plasma research to various instabilities, such as electrostatic drift wave turbulence and transport,[15,16] Alfven eigenmodes and energetic particle modes,[17] current-driven kink and tearing instabilities, and radio frequency (RF) waves.[18,19] Meanwhile, gyrokinetic codes[20–23] have served as an important type of simulation tool that has given a tremendous boost to research on plasma physics for both low-frequency processes[20,24] described by ion gyrokinetic theory and high-frequency processes[25,26] described by electron gyrokinetic theory.

For low-frequency electromagnetic fluctuations with short wavelengths perpendicular to the magnetic field, Vlasov-Maxwell equations can be used to construct a set of self-consistent gyrokinetic-Maxwell differential equation systems. This set usually consists of gyrokinetic Vlasov equations given in terms of Hamilton's equations in gyrocenter phase space and the gyrokinetic Maxwell equations or force-balance equations expressed in terms of moments[27] of the gyrocenter phase-space distribution. In this procedure, first, gyrocenter Hamilton's equations are derived from the gyrocenter Hamiltonian using the Lie-transform perturbation method,[10,13,28] which decouples complete particle dynamics into the fast gyromotion part and the slow gyrocenter drift motion part. Then, the gyrokinetic Maxwell equations are obtained through the conventional approach,[11–13,29] the purely pullback transformation approach,[30] or the purely pushforward transformation approach..[31–34] These three approaches are equivalent in principle.

In the purely pullback approach, the distribution function is transformed from gyrocenter phase space to guiding-center phase space and then to particle phase space. This approach has been used to construct the high-frequency simulation model[30] for RF waves, where electrons are treated with a gyrokinetic description and ions are treated with a fully kinetic description.

In the conventional approach, the distribution function is transformed from gyrocenter phase space to guiding-center phase space through a pullback transformation, whereas the velocity is transformed from particle phase space to guiding-center phase space through a pushforward transformation. With this approach, the gyrocenter polarization effect, which was first discovered by Lee,[35] can be extensively investigated in the electrostatic[10] and electromagnetic[29,36] cases.

On the other hand, in the purely pushforward transformation approach, particle velocity is first transformed from particle phase space to guiding-center phase space and subsequently transformed to gyrocenter phase space through a two-step pushforward transformation, where the guiding-center phase-space gyroradius $\boldsymbol{\rho}_u$ and the gyrocenter phase-space gyroradius $\boldsymbol{\rho}_y$ are introduced, giving rise to the

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polarization and magnetization effects. This approach has been used to derive the moments and to investigate the particle polarization flux in the electrostatic case.[32]

In this work, given electromagnetic perturbations and a local Maxwellian equilibrium distribution, the moments of the distribution with the finite Larmor radius (FLR) effect retained are derived through the purely pushforward transformation approach, which is used to construct our physical model and which is useful for both code development and analytic theory. Using the virtue of this approach, i.e., the very clear physical meaning of moments, we discuss the macroscopical equilibrium flow and perturbed flow.

The small parameters ϵ_B, ϵ_ω, $\epsilon_{||}$, and ϵ_δ are introduced to track the nonlinear gyrokinetic spatial-temporal orderings. First, the gyroradius $\rho = v_t/\Omega$ is small compared with the characteristic lengths L of the equilibrium profiles, such as the density, temperature, and magnetic field

$$\frac{\rho}{L} \sim \epsilon_B,$$

where v_t is the thermal velocity, $\Omega = (qB_0)/(mc)$ is the particle cyclotron frequency in an unperturbed magnetic field B_0, and q and m are the charge and mass of a particle, respectively. Second, the temporal ordering of the fluctuating fields satisfies

$$\frac{\omega}{\Omega} \sim \epsilon_\omega,$$

where ω is the characteristic frequency of the fluctuations. Additionally, the perpendicular and parallel spatial orderings of the fluctuating fields meet

$$k_\perp \rho \equiv \epsilon_\perp \sim 1 \quad \text{and} \quad k_{||}\rho \sim \epsilon_{||} \ll 1,$$

where $\epsilon_\perp \gg \epsilon_{||}$. The amplitude of the perturbed quantities is described by the small parameter ϵ_δ

$$\frac{\delta B}{B} \sim \frac{\delta f}{f} \sim \frac{q\delta\phi}{T} \sim \epsilon_\delta,$$

where δB is the amplitude of the perturbed magnetic field, $\delta\phi$ is the perturbed electrostatic potential, and δf is the perturbed distribution function. The relationships of these parameters depend on the characteristics of specific physical processes. These small parameters that appear ahead of physical quantities in the rest of this paper act as indexes that indicate the ordering of these quantities. Although they are treated approximately equal during the model derivation, i.e., $\epsilon_\omega \sim \epsilon_{||} \sim \epsilon_B \sim \epsilon_\delta \sim \epsilon$, their properties are retained.

The inventory of this paper is as follows: Section II reviews modern nonlinear gyrokinetic theory[14] and phase-space transformation. Section III presents the exact solution of the gauge scalar field S_1. In Sec. IV, with the preparation of the pushforward transformation, the gyrocenter phase-space gyroradius $\boldsymbol{\rho}_y$ is derived through the guiding-center gyroradius $\boldsymbol{\rho}_u$ and S_1. In Sec. V, with the purely pushforward transformation approach, the gyrocenter moments are presented. In Sec. VI, from the perspective of polarization and magnetization, the relationships between single-particle motion and macroscopic flow are analyzed. In Sec. VII, the results with a long wavelength limit are listed. In Sec. VIII, a discussion is given.

II. PERTURBED HAMILTONIAN DYNAMICS

According to modern nonlinear gyrokinetic theory,[14] a two-step procedure is used in the dynamic reduction of a single-particle Hamiltonian system to decouple the fast time scale gyromotion from the slow gyrocenter motions determined by the relevant electromagnetic field. A very efficient method for deriving the reduced Hamilton's equations is the Lie-transform perturbation method,[37,38] which is the foundation of modern nonlinear gyrokinetic theory. This method includes two-step near-identity transformations in extended phase space

$$T_\epsilon^{\pm 1} \equiv \exp\left(\pm \sum_{n=1}^{\infty} \epsilon^n \mathcal{L}_n\right),$$

where $\mathcal{L}_n$ is the nth-order Lie derivative generated by the nth-order generating vector field $\mathbf{G}_n$. The positive symbol denotes pullback transformation T_ϵ, and the negative symbol denotes pushforward transformation T_ϵ^{-1}.

With the careful choice of Hamiltonian representation, it is feasible to zero out the non-zero-order symplectic structure of the system Lagrangian in the extended gyrocenter phase space, $\bar{\Gamma}_n \equiv 0$ for $n > 0$, and

$$\bar{\Gamma}_0 = \left[\frac{q}{c}\mathbf{A}_0 + \bar{p}_{||}\hat{\mathbf{b}}\right] \cdot d\bar{\mathbf{X}} + \frac{\bar{\mu}B_0}{\Omega} d\bar{\theta} - \bar{w}dt,$$

where $\mathbf{A}_0$ is the unperturbed vector potential and $\hat{\mathbf{b}}$ is the unit vector of the unperturbed magnetic field $\mathbf{B}_0$. The extended gyrocenter phase-space coordinates $\bar{\mathcal{Z}}(\bar{\mathbf{X}}, \bar{p}_{||}, \bar{\mu}, \bar{\theta}, \bar{w}, t)$ are transformed from extended guiding-center phase-space coordinates $\mathcal{Z}(\mathbf{X}, p_{||}, \mu, \theta, w, t)$ via gyrocenter transformation, where $(w,\ t)$ are the canonically conjugate guiding-center energy-time coordinates. The guiding-center coordinates $\mathcal{Z}(\mathbf{X}, p_{||}, \mu, \theta, t)$ are obtained from particle phase-space coordinates $z(\mathbf{x}, p_{0||}, \mu_0, \theta_0, t)$ via guiding-center transformation. The particle phase-space coordinates are defined as follows: $\mathbf{x}$ is the particle position, $\mu_0 = mv_\perp^2/(2B_0)$ is the magnetic moment, θ_0 is the phase angle, and $p_{0||} = mv_{||}$ is the kinetic momentum parallel to the unperturbed magnetic field.

The gyrocenter Hamiltonian in extended gyrocenter phase space is

$$\bar{\mathcal{H}}_y = \frac{1}{2m}\bar{p}_{||}^2 + \bar{\mu}B_0 + \epsilon_\delta q\langle\delta\phi_u^*\rangle - \bar{w}.$$

The choice of Hamiltonian representation determines that the gyrocenter parallel momentum $\bar{p}_{||}$ is the canonical momentum $\bar{p}_{||} = p_{||} + q/c\delta A_{||}$. The effective potential

$$\delta\phi_u^* = \delta\phi_u - \frac{\delta\mathbf{A}_u}{c} \cdot \left(\frac{p_{||}}{m}\hat{\mathbf{b}} + \Omega\frac{\partial\boldsymbol{\rho}_u}{\partial\theta}\right)$$

includes the perturbed scalar potential $\delta\phi_u(\mathbf{X}, t)$ and vector potential $\delta\mathbf{A}_u(\mathbf{X}, t)$ in guiding-center phase space.

$\boldsymbol{\rho}_u = (2\mu B_0/m)^{1/2}\ \hat{\boldsymbol{\rho}}/\Omega$ is the guiding-center gyroradius derived by guiding-center phase-space transformation,[9,14] and $\hat{\boldsymbol{\rho}}$ is the basis vector of $\boldsymbol{\rho}_u$. Accordingly, the Poisson bracket for two arbitrary functions $\mathcal{F}$ and $\mathcal{G}$ in the extended gyrocenter phase space is defined as

$$\{\mathcal{F},\mathcal{G}\} = \frac{q}{mc}\left(\frac{\partial\mathcal{F}}{\partial\bar{\theta}}\frac{\partial\mathcal{G}}{\partial\bar{\mu}} - \frac{\partial\mathcal{F}}{\partial\bar{\mu}}\frac{\partial\mathcal{G}}{\partial\bar{\theta}}\right) + \frac{\mathbf{B}_0^*}{B_{||}^*}\cdot\left(\bar{\nabla}\mathcal{F}\frac{\partial\mathcal{G}}{\partial\bar{p}_{||}} - \frac{\partial\mathcal{F}}{\partial\bar{p}_{||}}\bar{\nabla}\mathcal{G}\right) - \frac{c\hat{\mathbf{b}}}{qB_{||}^*}\cdot\bar{\nabla}\mathcal{F}\times\bar{\nabla}\mathcal{G} + \left(\frac{\partial\mathcal{F}}{\partial\bar{w}}\frac{\partial\mathcal{G}}{\partial t} - \frac{\partial\mathcal{F}}{\partial t}\frac{\partial\mathcal{G}}{\partial\bar{w}}\right),$$

where $\mathbf{B}_0^*$ is the modified magnetic field

$$\mathbf{B}_0^* = \mathbf{B}_0 + \epsilon_B\frac{B_0\bar{p}_{||}}{m\Omega}\bar{\nabla}\times\hat{\mathbf{b}},$$

and $B_{||}^* = \hat{\mathbf{b}}\cdot\mathbf{B}_0^*$.

The gyrocenter Hamiltonian equations are

$$\dot{\bar{p}}_{||} = -\frac{\epsilon_{||}\epsilon_\delta}{\epsilon_B}q\frac{\mathbf{b}^*}{b_{||}^*}\cdot\bar{\nabla}\langle\delta\phi_u^*\rangle - \bar{\mu}\frac{\mathbf{b}^*}{b_{||}^*}\cdot\bar{\nabla}B_0, \tag{1}$$

$$\dot{\bar{\mathbf{X}}} = \left(\frac{\bar{p}_{||}}{m} + \epsilon_\delta\frac{\partial\langle q\delta\phi_u^*\rangle}{\partial\bar{p}_{||}}\right)\frac{\mathbf{b}^*}{b_{||}^*} + \epsilon_\delta\frac{c}{B_{||}^*}\hat{\mathbf{b}}\times\bar{\nabla}\langle\delta\phi_u^*\rangle + \epsilon_B\frac{c\bar{\mu}}{qB_{||}^*}\hat{\mathbf{b}}\times\bar{\nabla}B_0, \tag{2}$$

where $\mathbf{b}^* = \hat{\mathbf{b}} + \epsilon_B\bar{p}_{||}\bar{\nabla}\times\hat{\mathbf{b}}/(m\Omega)$ and $b_{||}^* = \hat{\mathbf{b}}\cdot\mathbf{b}^*$. The perturbed linear gyrocenter dynamics contained in Eq. (2) include the curvature drift velocity $\bar{p}_{||}^2\bar{\nabla}\times\hat{\mathbf{b}}/(m^2\Omega)$, the linear electrostatic perturbed $\mathbf{E}\times\mathbf{B}_0$ velocity $\delta\mathbf{v}_E = c\hat{\mathbf{b}}\times\bar{\nabla}\delta\phi/B_0$, the perturbed magnetic-flutter velocity $\bar{p}_{||}\delta\mathbf{B}_\perp/B_0$, and the perturbed grad-$\delta B_{||}$ velocity $c\hat{\mathbf{b}}\times\bar{\nabla}\delta B_{||}(qB_0)$. For convenience, these types of drifts, which are independent of the time derivative of perturbed fields, are denoted by $\mathbf{V}_D$. Due to the choice of the Hamiltonian gyrokinetic model, the absences in Eq. (2) are the drifts depending on the time derivative of perturbed fields, such as the polarization drift velocity $1/(\Omega B_0)d\mathbf{E}/dt$, the linear induced perturbed $\mathbf{E}\times\mathbf{B}_0$ drift velocity $c\hat{\mathbf{b}}/B_0\times\partial\delta\mathbf{A}/\partial t$, and the drift velocity related to $d\delta\mathbf{B}/dt$.

III. GAUGE FIELD OF GYROCENTER PHASE-SPACE TRANSFORMATION

The generating vector function $\mathbf{G}_n$ for the phase-space transformation is obtained by choosing a special symplectic form of the gyrocenter phase-space Lagrangian. As a result, it affects the choice of phase-space gauge function S_n that has no effect on the Poisson bracket structure. In the Hamiltonian gyrokinetic model, for example, the first-order generating vector field is

$$G_1^\alpha = \{S_1,\mathcal{Z}^\alpha\}_0 + \epsilon_\delta\frac{q}{c}\delta\mathbf{A}_u\cdot\{\mathbf{X}+\epsilon_B\boldsymbol{\rho}_u,\mathcal{Z}^\alpha\}.$$

The first-order gauge scalar field S_1 is chosen as $\{S_1,\mathcal{H}_{0u}\} = \epsilon_\delta q\delta\tilde{\phi}_u^*$ to ensure that the first-order gyrocenter Hamiltonian

$$\bar{H}_{1y} \equiv \epsilon_\delta q\delta\phi_u^* - \{S_1,\mathcal{H}_{0u}\} = \epsilon_\delta q\langle\delta\phi_u^*\rangle \tag{3}$$

is independent of the gyrocenter gyroangle, where $\mathcal{H}_{0u} = H_{0u} - w$ is the unperturbed extended guiding-center phase-space Hamiltonian and $\delta\tilde{\phi}_u^* \equiv \delta\phi_u^* - \langle\delta\phi_u^*\rangle$ is the gyroangle-dependent part of $\delta\phi_u^*$. According to the transformation of the first-order gyrocenter Hamiltonian (3), a possible solution of S_1[33] is

$$S_1 = \frac{q}{\Omega}\tilde{\Phi}_u^* - \epsilon_B\frac{q}{\Omega^2}\int\left(\left\{\tilde{\Phi}_u^*,\mathcal{H}_{0u}\right\} - \Omega\delta\tilde{\phi}_u^*\right)d\bar{\theta} + \epsilon_B^2\frac{q}{\Omega^3}\int\left[\left\{\int\left(\left\{\tilde{\Phi}_u^*,\mathcal{H}_{0u}\right\} - \Omega\delta\tilde{\phi}_u^*\right)d\bar{\theta},\mathcal{H}_{0u}\right\} - \Omega\left(\left\{\tilde{\Phi}_u^*,\mathcal{H}_{0u}\right\} - \Omega\delta\tilde{\phi}_u^*\right)\right]d\bar{\theta} + \cdots, \tag{4}$$

where $\tilde{\Phi}_u^* = \int\delta\tilde{\phi}_u^* d\bar{\theta}$, and its detailed expression is

$$\begin{aligned}\tilde{\Phi}_u^* = {} & \left(\sum_{-\infty}^{-1} + \sum_{1}^{\infty}\right)i^l e^{il\alpha}J_l\frac{\exp(il\theta)}{il}\left(\delta\phi - \frac{\bar{p}_{||}}{cm}\delta A_{||}\right) \\ & - \frac{1}{2c}\sqrt{\frac{2\bar{\mu}B_0}{m}}\left\{\left(\sum_{-\infty}^{-2} + \sum_{0}^{\infty}\right)i^l e^{il\alpha}J_l \times\left[\frac{\exp i(l+1)\theta}{l+1}\delta A_x - \frac{\exp i(l+1)\theta}{i(l+1)}\delta A_y\right]\right. \\ & \left. - \left(\sum_{-\infty}^{0} + \sum_{2}^{\infty}\right)i^l e^{il\alpha}J_l \times\left[\frac{\exp i(l-1)\theta}{l-1}\delta A_x + \frac{\exp i(l-1)\theta}{i(l-1)}\delta A_y\right]\right\}.\end{aligned} \tag{5}$$

The fundamental ordering of S_1 is $\epsilon_\delta\epsilon_B$ since $\{S_1,\mathcal{H}_{0u}\}$ is proportional to the first-order Hamiltonian.

In this work, two types of local coordinate systems are used as a generalization of a Euclidean space, which moves along the unperturbed curved magnetic field line:[39] One type is the local right-handed Cartesian coordinate system (x,y,z) with the unit basis vectors $(\hat{\mathbf{e}}_x,\hat{\mathbf{e}}_y,\hat{\mathbf{b}})$, and the other type is the rotating left-handed cylindrical coordinate system. The cylindrical coordinate system includes the guiding-center cylindrical coordinate system (ρ,θ,z) with the unit basis vectors $(\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\theta}},\hat{\mathbf{b}})$ and the gyrocenter cylindrical coordinate system $(\bar{\rho},\bar{\theta},z)$ with the unit basis vectors $(\hat{\bar{\boldsymbol{\rho}}},\hat{\bar{\boldsymbol{\theta}}},\hat{\mathbf{b}})$, and the relationships between their basis vectors and Cartesian basis vectors are

$$\hat{\boldsymbol{\rho}} = \cos\theta\hat{\mathbf{e}}_x - \sin\theta\hat{\mathbf{e}}_y,\quad \hat{\boldsymbol{\theta}} = \frac{\partial\hat{\boldsymbol{\rho}}}{\partial\theta},$$
$$\hat{\bar{\boldsymbol{\rho}}} = \cos\bar{\theta}\hat{\mathbf{e}}_x - \sin\bar{\theta}\hat{\mathbf{e}}_y,\quad \hat{\bar{\boldsymbol{\theta}}} = \frac{\partial\hat{\bar{\boldsymbol{\rho}}}}{\partial\bar{\theta}}.$$

Incidentally, according to the appendix in Ref. 6, Littlejohn's gyrogauge vector field $\mathbf{R} = \nabla\hat{\mathbf{e}}_x\cdot\hat{\mathbf{e}}_y$, which represents the spatial dependence of a perpendicular basis vector, would modify the operator ∇ in the Poisson bracket structure but has no effect on Hamilton's equations, and the modification of the gyrocenter gyroradius $\boldsymbol{\rho}_y$ in this work is on the high order. Therefore, the vector field $\mathbf{R}$ is directly neglected in this work. Moreover, α denotes the angle between the wave vector $\mathbf{k}$ and $\hat{\mathbf{e}}_x$ in Eq. (5).

IV. GYROCENTER GYRORADIUS

The gyroradius $\boldsymbol{\rho}_y$ in the gyrocenter phase space is necessary for the pushforward transformation. It is the distance from the particle position $c(z)=\mathbf{x}$ to the gyrocenter position $c(\bar{\mathcal{Z}})=\bar{\mathbf{X}}$ under a total electromagnetic field, whereas the guiding-center gyroradius $\boldsymbol{\rho}_u$ is the distance from the particle position $c(z)=\mathbf{x}$ to the guiding-center position $c(\mathcal{Z})=\mathbf{X}$ under an equilibrium magnetic field, where the function c is defined to choose the position coordinate. They satisfy the equation

$$\boldsymbol{\rho}_u(\mathcal{Z})+c(\mathcal{Z})\equiv\boldsymbol{\rho}_y(\bar{\mathcal{Z}})+c(\bar{\mathcal{Z}}).$$

When $\boldsymbol{\rho}_u$ and $\boldsymbol{\rho}_y$ are compared in gyrocenter phase space, it is found that they possess the relationship

$$\begin{aligned}\boldsymbol{\rho}_y(\bar{\mathcal{Z}})&\equiv T_y^{-1}\left[c(\bar{\mathcal{Z}})+\epsilon_B\boldsymbol{\rho}_u(\bar{\mathcal{Z}})\right]-c(\bar{\mathcal{Z}})\\&=\epsilon_B\boldsymbol{\rho}_u-\epsilon_\delta\mathbf{G}_1\cdot\left(\bar{\mathbf{X}}+\epsilon_B\boldsymbol{\rho}_u\right),\end{aligned}$$

where the guiding-center phase-space gyroradius $\boldsymbol{\rho}_u(\mathcal{Z})$ has been expressed by the gyrocenter phase-space coordinate $\boldsymbol{\rho}_u(\bar{\mathcal{Z}})$. With the detailed expressions of the first-order gauge scalar field (4) and the gyroangle-dependent part of the effective potential (5), $\boldsymbol{\rho}_y$ becomes

$$\begin{aligned}\boldsymbol{\rho}_y&=\epsilon_B\boldsymbol{\rho}_u-\epsilon_B\epsilon_\delta\frac{q}{mc}\left(\frac{\partial S_1}{\partial\bar{\theta}}\frac{\partial\boldsymbol{\rho}_u}{\partial\bar{\mu}}-\frac{\partial S_1}{\partial\bar{\mu}}\frac{\partial\boldsymbol{\rho}_u}{\partial\bar{\theta}}\right)+\epsilon_B\epsilon_\delta\frac{\partial S_1}{\partial\bar{p}_{||}}\frac{\mathbf{B}_0^*}{B_{||}^*}\\&=\epsilon_B\boldsymbol{\rho}_u-\epsilon_B\epsilon_\delta\frac{q}{B_0\Omega}\left(\delta\tilde{\phi}_u^*\sqrt{\frac{B_0}{2m\bar{\mu}}}\hat{\bar{\boldsymbol{\rho}}}-\sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\theta}}}\right)\\&\quad+\epsilon_B\epsilon_\delta\frac{q}{\Omega}\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{p}_{||}}\frac{\mathbf{B}_0^*}{B_{||}^*},\end{aligned}\tag{6}$$

and the gyroaveraged part is

$$\begin{aligned}\langle\boldsymbol{\rho}_y\rangle=&-\frac{q}{B_0\Omega}\left[iJ_1\sqrt{\frac{B_0}{2m\bar{\mu}}}+\frac{iB_0k_\perp}{2m\Omega}(J_0-J_2)\right]\\&\times\left(\cos\alpha\hat{\mathbf{e}}_x+\sin\alpha\hat{\mathbf{e}}_y\right)\left(\delta\phi-\frac{p_{||}}{cm}\delta A_{||}\right)-\frac{J_0}{B_0}\delta\mathbf{A}\times\hat{\mathbf{b}}\\&+\frac{k_\perp}{4B_0\Omega}\sqrt{\frac{2\bar{\mu}B_0}{m}}\delta\mathbf{A}\cdot\left\{\left[\sin2\alpha\left(\hat{\mathbf{e}}_y\hat{\mathbf{e}}_y-\hat{\mathbf{e}}_x\hat{\mathbf{e}}_x\right)\right.\right.\\&\left.\left.+\cos2\alpha\left(\hat{\mathbf{e}}_y\hat{\mathbf{e}}_x+\hat{\mathbf{e}}_x\hat{\mathbf{e}}_y\right)\right](J_1-J_3)-2J_1\left(\hat{\mathbf{e}}_x\hat{\mathbf{e}}_y-\hat{\mathbf{e}}_y\hat{\mathbf{e}}_x\right)\right\},\end{aligned}\tag{7}$$

where J_n is the nth-order Bessel function. The nonzero of $\langle\boldsymbol{\rho}_y\rangle$ implies that part of the drift motion is included in the gyroradius. Next, the particle velocity $d\mathbf{x}/dt$ from the guiding-center and gyrocenter phase space is investigated in order to analyze the physical meaning of Eq. (7).

In the absence of perturbed electromagnetic fluctuations, a guiding-center transformation is sufficient. The transformed velocity

$$T_\epsilon^{-1}\mathbf{v}=\frac{d_\epsilon\mathbf{X}}{dt}+\frac{d_\epsilon\boldsymbol{\rho}_u}{dt}$$

is decoupled into guiding-center motion

$$\frac{d_\epsilon\mathbf{X}}{dt}=\frac{p_{||}}{m}\frac{\mathbf{b}^*}{b_{||}^*}+\epsilon_B\frac{c\mu}{qB_{||}^*}\hat{\mathbf{b}}\times\nabla B_0$$

and particle polarization motion

$$\frac{d_\epsilon\boldsymbol{\rho}_u}{dt}=\sqrt{\frac{2\mu B_0}{m}}\hat{\boldsymbol{\theta}}.$$

It indicates that the gradient of B_0 produces the guiding-center drift velocity. $\mathbf{v}_\perp=(2\mu B_0/m)^{1/2}\hat{\boldsymbol{\theta}}$ is the Larmor cyclotron velocity in guiding-center phase space. In the presence of perturbed electromagnetic fluctuations, $d\mathbf{x}/dt$ should be transformed into gyrocenter phase space

$$T_\epsilon^{-1}\mathbf{v}=\frac{d_\epsilon\bar{\mathbf{X}}}{dt}+\frac{d_\epsilon\boldsymbol{\rho}_y}{dt},\tag{8}$$

where $d_\epsilon\bar{\mathbf{X}}/dt$ denotes the gyrocenter drift velocity, which is given by Hamilton's Equation (2), and the particle polarization velocity $d_\epsilon\boldsymbol{\rho}_y/dt$[40] is

$$\begin{aligned}\frac{d_\epsilon\boldsymbol{\rho}_y}{dt}=&\epsilon_\omega\epsilon_\delta\frac{\partial\boldsymbol{\rho}_y}{\partial t}+\left\{\boldsymbol{\rho}_y,\bar{H}_y\right\}\\=&\epsilon_\omega\epsilon_\delta\frac{\partial\boldsymbol{\rho}_y}{\partial t}+\sqrt{\frac{2\bar{\mu}B_0}{m}}\hat{\bar{\boldsymbol{\theta}}}\left(1+\epsilon_\delta\frac{q}{B_0}\frac{\partial\langle\delta\phi_u^*\rangle}{\partial\bar{\mu}}\right)\\&-\epsilon_\delta\frac{q}{B_0}\frac{\partial}{\partial\bar{\theta}}\left(\delta\tilde{\phi}_u^*\sqrt{\frac{B_0}{2m\bar{\mu}}}\hat{\bar{\boldsymbol{\rho}}}-\sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\theta}}}-\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{p}_{||}}\frac{B_0\mathbf{B}_0^*}{B_{||}^*}\right)\\&+\epsilon_B\left(\frac{\bar{p}_{||}}{m}+\epsilon_\delta\frac{\partial\langle\delta\phi_u^*\rangle}{\partial\bar{p}_{||}}\right)\frac{\mathbf{b}^*}{b_{||}^*}\cdot\bar{\nabla}\boldsymbol{\rho}_y+\epsilon_B\epsilon_\delta\frac{c}{B_{||}^*}\left(\hat{\mathbf{b}}\times\bar{\nabla}\langle\delta\phi_u^*\rangle\right)\\&\cdot\bar{\nabla}\boldsymbol{\rho}_y+\epsilon_B^2\frac{c\bar{\mu}}{qB_{||}^*}\left(\hat{\mathbf{b}}\times\bar{\nabla}B_0\right)\cdot\bar{\nabla}\boldsymbol{\rho}_y.\end{aligned}\tag{9}$$

According to the transformation of $d\mathbf{x}/dt$, Eq. (9) shows that $\langle d\boldsymbol{\rho}_y/dt\rangle$ undertakes the drifts related to the time derivative of perturbed fields. In addition, it indicates that the perturbed gyrocenter Hamiltonian $\bar{H}_{gy1}$ modifies Larmor cyclotron motion.

Therefore, the slow time-scale drift motion is not fully decoupled from particle motion in the Hamiltonian gyrokinetic model; instead, only $\mathbf{V}_D$ emerges in gyrocenter motion $\dot{\bar{\mathbf{X}}}$. In this way, the particle motion can be described such that $\mathbf{V}_D$ pushes the gyrocenter position, and the charged particle performs the cyclotron motion accompanied by drift motion at the speed of $\langle d\boldsymbol{\rho}_y/dt\rangle$ in the gyrocenter reference frame.

V. MOMENTS OF GYROCENTER DISTRIBUTION

A closed self-consistent description of the interactions involving the perturbed electromagnetic field and a gyrocenter Vlasov distribution implies that the gyrokinetic Maxwell equations are written with moments expressed in terms of the gyrocenter distribution function. There are three approaches to calculating the moments: the conventional approach, the purely pullback transformation approach, and the purely pushforward transformation approach.

According to the character of the Lie transform, these three approaches are identical. For specific distributions, such as the Maxwellian distribution, the three approaches have their own advantages and disadvantages. The calculations for the pullback transformation approach and the conventional approach are simpler than those for the purely pushforward transformation approach since the integral of the transformed distribution over the particle phase space is simpler than that of the transformed velocity over the gyrocenter phase space, especially for high-order moments and high-order transformations. Furthermore, they can avoid the detailed expression of $\boldsymbol{\rho}_y$. In this work, the purely pushforward transformation approach will be used, by which the polarization and magnetization of the gyrocenter[41] can be visually revealed.

The equilibrium distribution is taken as a Maxwellian, $\bar{F}_0 = \bar{N}_0 \bar{F}_M$, where $\bar{F}_M$ is the normalized Maxwellian distribution. Then, the moment equation of the distribution function reads

$$\begin{aligned}\Lambda(\mathbf{r}) &= \int g(\mathbf{v}) f(\mathbf{x},\mathbf{v})\delta(\mathbf{x}-\mathbf{r})d\mathbf{x}d\mathbf{v}\\ &= \int g\left(T_\epsilon^{-1}\mathbf{v}\right)\bar{F}(\bar{\mathcal{Z}})\delta\left(\bar{\mathbf{X}}+\boldsymbol{\rho}_y-\mathbf{r}\right)d\bar{p}^3\\ &= \sum_0^\infty (-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\int \bar{F}\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\,g\left(T_\epsilon^{-1}\mathbf{v}\right)d\bar{p}^3, \quad (10)\end{aligned}$$

where $\int d^3\bar{p} = \int B_{||}^*/(m^2)d\bar{\mu}d\bar{p}_{||}d\bar{\theta}$ denotes the integrals over the gyrocenter phase-space canonical momentum $\bar{p}_{||}$, phase angle $\bar{\theta}$, and magnetic moment $\bar{\mu}$. For the purely pushforward approach, the particle phase-space velocity is transformed using a pushforward transformation, and the Dirac delta function is expanded using the gyrocenter phase-space gyroradius $\boldsymbol{\rho}_y$. Nevertheless, for the conventional approach, the gyrocenter phase-space distribution is transformed through a pullback transformation, and the Dirac delta function is expanded through $\boldsymbol{\rho}_u$.

For the integral of Eq. (10), taking the full FLR effect into consideration is a challenge. However, in this work, only the linear moments are considered, and terms in moment equations that are higher than ϵ are neglected; thus, the calculation difficulties can be settled without ignoring the FLR effect

$$\begin{aligned}\Lambda(\mathbf{r}) = &\sum_0^1 (-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\int \bar{F}_0\,\overbrace{\boldsymbol{\rho}_u\cdots\boldsymbol{\rho}_u}^{n}\,g_0 d\bar{p}^3\\ &+\epsilon_\delta\int \exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\bar{F}_1 g_0 d\bar{p}^3\\ &-\epsilon_\delta\bar{\nabla}\cdot\int \bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\boldsymbol{\rho}_y' g_0 d\bar{p}^3\\ &+\epsilon_\delta\int \bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)g_1 d\bar{p}^3, \quad (11)\end{aligned}$$

where $\epsilon_\delta\epsilon_B\boldsymbol{\rho}_y' = \epsilon_B\boldsymbol{\rho}_y - \epsilon_B\boldsymbol{\rho}_u$ is the perturbed part of the gyrocenter gyroradius and the subscripts 0 and 1 stand for the unperturbed and perturbed quantities, respectively. By setting $g(\mathbf{v})$ equal to 1, $qT_\epsilon^{-1}\mathbf{v}$, and $mT_\epsilon^{-1}\mathbf{v}T_\epsilon^{-1}\mathbf{v}$, the particle density, current density, and pressure tensor, respectively, can be derived as follows:

$$\begin{aligned}n(\mathbf{r}) = &\bar{N}_0 + \epsilon_\delta\frac{q\bar{N}_0}{\bar{T}}\delta\phi\left(\langle J_0^2\rangle_{\bar{p}}-1\right)\\ &-\epsilon_\delta\frac{q\bar{N}_0}{c\bar{T}}\left\langle\sqrt{\frac{2\bar{\mu}B_0}{m}}iJ_0J_1\right\rangle_{\bar{p}}\frac{\mathbf{k}_\perp\times\hat{\mathbf{b}}}{k_\perp}\cdot\delta\mathbf{A}_\perp\\ &+\epsilon_\delta\int\bar{F}_1J_0d^3\bar{p}, \quad (12)\end{aligned}$$

$$\begin{aligned}\mathbf{J}(\mathbf{r}) = &\epsilon_B\frac{\hat{\mathbf{b}}}{B_0}\times\bar{\nabla}(c\bar{N}_0\bar{T}) - \epsilon_\delta\frac{q^2\bar{N}_0}{cm}\delta A_{||}\langle J_0^2\rangle_{\bar{p}}\hat{\mathbf{b}}\\ &+\epsilon_\delta\frac{q^2\bar{N}_0}{\bar{T}}\left\langle\sqrt{\frac{2\bar{\mu}B_0}{m}}iJ_1\frac{\hat{\mathbf{b}}\times\mathbf{k}_\perp}{k_\perp}\langle\delta\phi_u^*\rangle\right\rangle_{\bar{p}}\\ &+\epsilon_\delta\int q\bar{F}_1\left[\frac{\bar{p}_{||}}{m}J_0\hat{\mathbf{b}} - iJ_1\frac{\mathbf{k}_\perp\times\hat{\mathbf{b}}}{k_\perp}\sqrt{\frac{2\bar{\mu}B_0}{m}}\right]d^3\bar{p}, \quad (13)\end{aligned}$$

$$\begin{aligned}\mathbf{P}(\mathbf{r}) = &\bar{N}_0\bar{T}\mathbf{I} + \epsilon_\delta\int\left[\frac{\bar{p}_{||}^2}{m}J_0\hat{\mathbf{b}}\hat{\mathbf{b}} + \bar{\mu}B_0(J_0+J_2)(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\right]\bar{F}_1d^3\bar{p}\\ &+\epsilon_\delta\frac{qB_0\bar{N}_0}{\bar{T}}\langle\bar{\mu}\left[(J_0+J_2)\langle\delta\phi_u^*\rangle-\delta\phi\right](\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\rangle_{\bar{p}}, \quad (14)\end{aligned}$$

where $\bar{N}_0 = \int\bar{F}_0d^3\bar{p}$ and $\langle\rangle_{\bar{p}} = \int\bar{F}_Md^3\bar{p}$. In Eq. (13), the drift currents related to the time derivative of perturbed fields are on the order of $\epsilon_\delta\epsilon_\omega$ (9) and are neglected here, $\hat{\mathbf{b}}\times\bar{\nabla}(c\bar{N}_0\bar{T})/B_0$ is the diamagnetic current, and the current $(\bar{N}_0q^2\langle J_0^2\rangle_{\bar{p}}\delta A_{||}\hat{\mathbf{b}})/(cm)$ is caused by the choice of the Hamiltonian gyrokinetic model. In the pressure tensor (14), the off-diagonal components related to $\hat{\mathbf{b}}$ and terms whose divergence is a higher-order contribution are neglected.

With the Coulomb gauge $\nabla\cdot\mathbf{A}=0$, the gyrokinetic Poisson's equation and Ampere's law are obtained

$$\bar{\nabla}_\perp^2\delta\phi = -4\pi\sum_\alpha q_\alpha\delta n_\alpha, \quad (15)$$

$$-\bar{\nabla}_\perp^2\delta\mathbf{A} = \frac{4\pi}{c}\sum_\alpha\delta\mathbf{J}_\alpha, \quad (16)$$

where α denotes the particle species.

VI. MICROSCOPIC FLOW AND MACROSCOPIC FLOW

Equation (13) embodies the relationship between single-particle drift motion and macroscopic flow. The drift motion proportional to particle canonical momentum $\bar{p}_{||}$ does not produce an equilibrium current. The diamagnetic drift velocity vanished in Hamilton's equations because Hamilton's equations are derived by single-particle Hamiltonian theory. The appearance of the diamagnetic current, the disappearance of the currents produced by curvature drift and grad-B_0 drift, and the perturbed currents all are attributed to the polarization and magnetization effects of the gyrocenter gyroradius.

Given the definition of the gyrocenter polarization vector

$$\mathbf{P}_\epsilon = -q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^{n-1}\int\bar{F}\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\,d\bar{p}^3 \quad (17)$$

using the Vlasov equation $\{\bar{F},\bar{\mathcal{H}}_y\}$ and the gyrocenter Liouville theorem

$$\frac{\partial B_{\|}^{*}}{\partial t}+\bar{\nabla}\cdot\left(B_{\|}^{*}\dot{\bar{\mathbf{X}}}\right)+\frac{\partial}{\partial\bar{p}_{\|}}\left(B_{\|}^{*}\dot{\bar{p}}_{\|}\right)=0,$$

the current density can be reformed as[34] (refer to the Appendix B)

$$\mathbf{J}(\mathbf{r})=\bar{\mathbf{J}}_{y}+\mathbf{J}_{p}+\mathbf{J}_{m}, \tag{18}$$

where

$$\begin{aligned}\bar{\mathbf{J}}_{y}=&\int q\dot{\bar{\mathbf{X}}}\bar{F}d^{3}\bar{p}=\epsilon_{B}\frac{c\bar{N}_{0}\bar{T}}{B_{0}}\bar{\nabla}\times\hat{\mathbf{b}}+\epsilon_{B}\frac{c\bar{N}_{0}\bar{T}}{B_{0}^{2}}\hat{\mathbf{b}}\times\bar{\nabla}B_{0}\\&+\epsilon_{\delta}q\bar{N}_{0}\left\langle-\frac{q}{cm}\delta A_{\|}J_{0}\hat{\mathbf{b}}+\frac{c}{B_{\|}^{*}}\hat{\mathbf{b}}\times\bar{\nabla}\langle\delta\phi_{u}^{*}\rangle\right\rangle_{\bar{p}}\\&+\epsilon_{\delta}\int q\bar{F}_{1}\frac{\bar{p}_{\|}}{m}\hat{\mathbf{b}}d^{3}\bar{p}\end{aligned} \tag{19}$$

is the gyrocenter drift current density,

$$\mathbf{J}_{p}=\frac{\partial\mathbf{P}_{\epsilon}}{\partial t}\sim\mathcal{O}(\epsilon^{2}) \tag{20}$$

is the polarization current, and

$$\begin{aligned}\mathbf{J}_{m}=&c\bar{\nabla}\times\mathbf{M}_{\epsilon}=-\epsilon_{B}\bar{\nabla}\times\left(\frac{c\bar{N}_{0}\bar{T}}{B_{0}}\hat{\mathbf{b}}\right)\\&-\epsilon_{\delta}\frac{q^{2}\bar{N}_{0}}{cm}\delta A_{\|}\langle J_{0}(J_{0}-1)\rangle_{\bar{p}}\hat{\mathbf{b}}-\epsilon_{\delta}\frac{cq\bar{N}_{0}}{B_{\|}^{*}}\hat{\mathbf{b}}\times\bar{\nabla}\langle\langle\delta\phi_{u}^{*}\rangle\rangle_{\bar{p}}\\&+\epsilon_{\delta}\frac{q^{2}\bar{N}_{0}}{\bar{T}}\left\langle\sqrt{\frac{2\bar{\mu}B_{0}}{m}}iJ_{1}\frac{\hat{\mathbf{b}}\times\mathbf{k}_{\perp}}{k_{\perp}}\langle\delta\phi_{u}^{*}\rangle\right\rangle_{\bar{p}}\\&+\epsilon_{\delta}\int q\bar{F}_{1}\left[\frac{\bar{p}_{\|}}{m}(J_{0}-1)\hat{\mathbf{b}}-iJ_{1}\frac{\mathbf{k}_{\perp}\times\hat{\mathbf{b}}}{k_{\perp}}\sqrt{\frac{2\bar{\mu}B_{0}}{m}}\right]d^{3}\bar{p}\end{aligned} \tag{21}$$

is the divergence-free magnetization current. The magnetization vector $\mathbf{M}_{\epsilon}$[42] reads

$$\begin{aligned}\mathbf{M}_{\epsilon}=&\epsilon_{\delta}\frac{q}{c}\int\sum_{1}^{\infty}\frac{1}{n!}\left(-\boldsymbol{\rho}_{u}\cdot\bar{\nabla}\right)^{n-1}\bar{F}_{0}\left(\boldsymbol{\rho}_{u}\times\dot{\bar{\mathbf{X}}}_{1}\right)d\bar{p}^{3}\\&+\epsilon_{\delta}\frac{q}{c}\int\sum_{1}^{\infty}\frac{1}{n!}\left(-\boldsymbol{\rho}_{u}\cdot\bar{\nabla}\right)^{n-1}\bar{F}_{1}\left(\boldsymbol{\rho}_{u}\times\dot{\bar{\mathbf{X}}}_{0}\right)d\bar{p}^{3}\\&+\epsilon_{\delta}\frac{q}{c}\int\sum_{1}^{\infty}\frac{1}{(n-1)!}\bar{F}_{0}\left(-\boldsymbol{\rho}_{u}\cdot\bar{\nabla}\right)^{n-1}\left(\boldsymbol{\rho}_{y}'\times\dot{\bar{\mathbf{X}}}_{0}\right)d\bar{p}^{3}\\&+\frac{q}{2c}\int\bar{F}_{0}\boldsymbol{\rho}_{u}\times\left(\dot{\boldsymbol{\rho}}_{y}\right)_{0}d\bar{p}^{3}\\&+\epsilon_{\delta}\frac{q}{c}\sum_{1}^{\infty}\frac{n}{(n+1)!}\int\left(-\boldsymbol{\rho}_{u}\cdot\bar{\nabla}\right)^{n-1}\bar{F}_{1}\boldsymbol{\rho}_{u}\times\left(\dot{\boldsymbol{\rho}}_{y}\right)_{0}d\bar{p}^{3}\\&+\epsilon_{\delta}\frac{q}{c}\sum_{1}^{\infty}\frac{n}{(n+1)!}\int\left(-\boldsymbol{\rho}_{u}\cdot\bar{\nabla}\right)^{n-1}\bar{F}_{0}\boldsymbol{\rho}_{u}\times\left(\dot{\boldsymbol{\rho}}_{y}\right)_{1}d\bar{p}^{3}\\&+\epsilon_{\delta}\frac{q}{c}\int\sum_{1}^{\infty}\frac{n^{2}}{(n+1)!}\left(-\boldsymbol{\rho}_{u}\cdot\bar{\nabla}\right)^{n-1}\bar{F}_{0}\left[\boldsymbol{\rho}_{y}'\times\left(\dot{\boldsymbol{\rho}}_{y}\right)_{0}\right]d\bar{p}^{3}\\&+\epsilon_{\delta}\frac{q}{c}\int\sum_{2}^{\infty}\frac{n^{2}-n}{(n+1)!}\left(-\boldsymbol{\rho}_{u}\cdot\bar{\nabla}\right)^{n-2}\bar{F}_{0}\\&\times\left\{\left[\left(\dot{\boldsymbol{\rho}}_{y}\right)_{0}\cdot\bar{\nabla}\right]\boldsymbol{\rho}_{y}'\times\boldsymbol{\rho}_{u}\right\}d\bar{p}^{3}.\end{aligned} \tag{22}$$

Obviously, Eq. (18) has the same result as Eq. (13). From the perspective of polarization (17) and magnetization (22), the relationship between microscopic flow and macroscopic flow is clear. For the macroscopic equilibrium flow, the gyrocenter magnetization produces the diamagnetic current and provides a current to cancel out the curvature and grad-B_0 drift current. Spitzer[43] first discussed this problem with the guiding-center motion equation and magnetohydrodynamic (MHD) equation, and Qin[44] discussed this problem with the gyrokinetic model. However, the physical pictures of the canceling of curvature and grad-B_0 drift current, along with the generation of macroscopic perturbed flow, are wanting. All of them will be presented in this work.

If the charged particles are positive, the trajectories of particles that move helically along field lines are left-handed from the view of the equilibrium magnetic field direction, as shown in Fig. 1. The current produced by the cyclotron motion of a particle can be treated as a small current coil. It can be seen that the number of these small current coils chained by boundary L increases with the gyroradius and particle number density. In this way, the gap between the current in the higher density and temperature areas and that in the lower density and temperature areas produces the diamagnetic current through the surface S surrounded by the boundary L.

When only the curve of the unperturbed magnetic field is taken into account, the number of small current coils chained by boundary L increases along the direction of curvature $\boldsymbol{\kappa}$, as shown in Fig. 2. Through the surface S in Fig. 2, the outward current at the upper left exceeds the inward current at the lower right, and then, a net outward current is produced. The direction of this current is exactly opposite to the gyrocenter drift current produced by the curvature of the magnetic field, and they cancel each other out.

When only the inhomogeneity of the unperturbed magnetic field is considered, as shown in Fig. 3, the Larmor gyroradius is smaller at the strong field site. Thus, the number of small current coils chained by boundary L at the weak field site is larger. In this way, through the surface S in Fig. 3, the outward current at the left exceeds the inward current at the

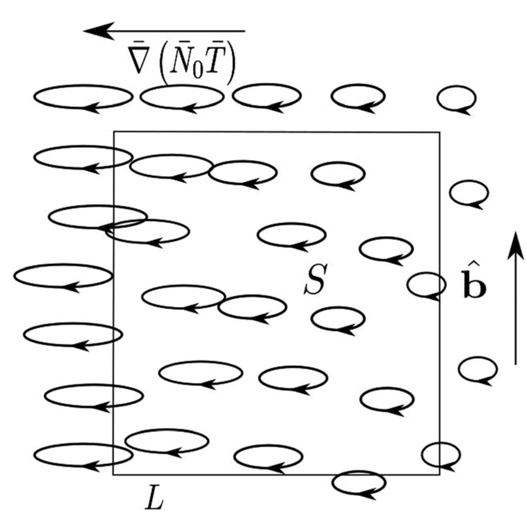

FIG. 1. Particle current coils in a uniform equilibrium magnetic field with nonuniform pressure.

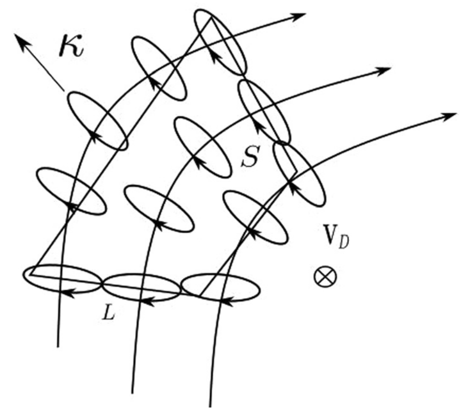

FIG. 2. Particle current coils in a curved equilibrium magnetic field.

right, and then, a net outward current is produced. The direction of this current is exactly opposite to the unperturbed grad-B_0 drift current, and they cancel each other out.

For the generation of macroscopic perturbed flow, the gyrocenter drift current originally holds the FLR effect, and the contribution of the polarization current is a higher-order effect. Comparison of the drift current with the polarization and magnetization currents indicates that the magnetization currents are equivalent to adding an extra FLR effect on the gyrocenter drift current, in contrast to the counterpart of the macroscopic equilibrium flow, where the magnetization current creates new flow and cancels out the gyrocenter drift flow.

VII. LONG WAVELENGTH LIMIT

Generally, a gyrokinetic system in a long wavelength limit is widely used and adequate for most physical problems, and the long wavelength limit is in accordance with the physical picture in the general case. In this limit, the gyroaveraged gyroradius (7) becomes

$$\langle \boldsymbol{\rho}_y \rangle = -\frac{1}{B_0}\left(\frac{c}{\Omega}\bar{\nabla}_\perp \delta\phi + \frac{\bar{p}_{||}}{m\Omega}\delta\mathbf{B}\times\hat{\mathbf{b}} + \delta\mathbf{A}\times\hat{\mathbf{b}}\right), \tag{23}$$

and the meaning of each term on the right of Eq. (23) is as follows: The first term is caused by the polarization drift

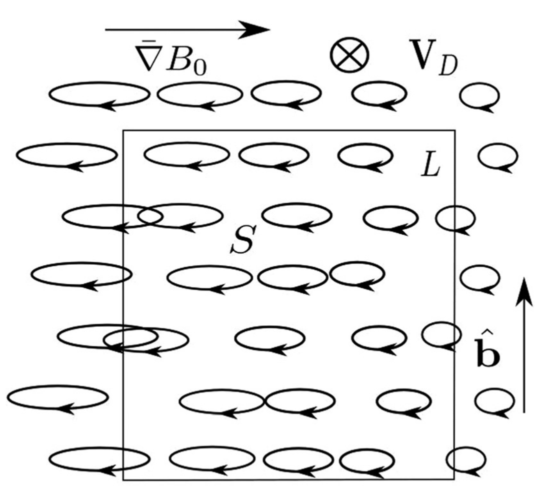

FIG. 3. Particle current coils in a nonuniform equilibrium magnetic field.

velocity $(1/\Omega B_0)d\mathbf{E}/dt$, the second term is caused by the drift related to $d\delta\mathbf{B}/dt$, and the last term is caused by the linear induced perturbed $\mathbf{E}\times\mathbf{B}_0$ drift velocity.

Similarly, the moment equations, gyrocenter drift current, and magnetization current can be reduced to

$$n = \bar{N}_0 + \epsilon_\delta\left[\bar{\nabla}\cdot\left(\frac{c\bar{N}_0}{B_0\Omega}\bar{\nabla}_\perp\delta\phi\right) + \frac{\bar{N}_0\delta B_{||}}{B_0}\right] + \epsilon_\delta\int \bar{F}_1 d^3\bar{p}, \tag{24}$$

$$\begin{aligned}\mathbf{J} = &-\epsilon_\delta\frac{\bar{N}_0 q^2}{cm}\delta A_{||}\hat{\mathbf{b}} + \epsilon_\delta\int q\frac{\bar{p}_{||}}{m}\hat{\mathbf{b}}\bar{F}_1 d^3\bar{p} + \epsilon_\delta\frac{cq\bar{N}_0}{B_0}\hat{\mathbf{b}}\\ &\times\bar{\nabla}\delta\phi + \epsilon_\delta\frac{3c^2\bar{N}_0\bar{T}}{2B_0^2\Omega}\hat{\mathbf{b}}\times\bar{\nabla}\bar{\nabla}_\perp^2\delta\phi + \epsilon_\delta\frac{2c\bar{N}_0\bar{T}}{B_0^2}\hat{\mathbf{b}}\\ &\times\bar{\nabla}\delta B_{||} + \epsilon_B\frac{\hat{\mathbf{b}}}{B_0}\times\bar{\nabla}(c\bar{N}_0\bar{T}) + \hat{\mathbf{b}}\times\bar{\nabla}\int c\bar{\mu}\bar{F}_1 d^3\bar{p},\end{aligned} \tag{25}$$

$$\begin{aligned}\mathbf{P} = &\bar{N}_0\bar{T}\mathbf{I} + \epsilon_\delta\frac{\bar{N}_0\bar{T}}{B_0}\left[2\delta B_{||}(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}}) + \frac{3}{2}\frac{c}{\Omega}\bar{\nabla}_\perp^2\delta\phi(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\right]\\ &+\epsilon_\delta\int\left[\frac{\bar{p}_{||}^2}{m}\hat{\mathbf{b}}\hat{\mathbf{b}} + \bar{\mu}B_0(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\right]\bar{F}_1 d^3\bar{p}.\end{aligned} \tag{26}$$

$$\begin{aligned}\bar{\mathbf{J}}_y = &-\epsilon_\delta\frac{\bar{N}_0 q^2}{cm}\delta A_{||}\hat{\mathbf{b}} + \epsilon_\delta\frac{c^2\bar{N}_0\bar{T}}{2B_0^2\Omega}\hat{\mathbf{b}}\times\bar{\nabla}\bar{\nabla}_\perp^2\delta\phi\\ &+\epsilon_B\frac{c\bar{N}_0\bar{T}}{B_0}\bar{\nabla}\times\hat{\mathbf{b}} + \epsilon_B\frac{c\bar{N}_0}{B_0}\hat{\mathbf{b}}\times\left[\frac{\epsilon_\delta}{\epsilon_B}\left(q\bar{\nabla}\delta\phi + \frac{\bar{T}}{B_0}\bar{\nabla}\delta B_{||}\right)\right.\\ &\left.+\frac{\bar{T}}{B_0}\bar{\nabla}B_0\right] + \epsilon_\delta\int q\frac{\bar{p}_{||}}{m}\mathbf{b}_0\bar{F}_1 d^3\bar{p},\end{aligned} \tag{27}$$

$$\begin{aligned}\mathbf{J}_m = &\epsilon_B\hat{\mathbf{b}}\times\bar{\nabla}\frac{c\bar{N}_0\bar{T}}{B_0} - \epsilon_B\frac{c\bar{N}_0\bar{T}}{B_0}\bar{\nabla}\times\hat{\mathbf{b}} + \epsilon_\delta\frac{c\bar{N}_0\bar{T}}{B_0}\hat{\mathbf{b}}\times\bar{\nabla}\frac{\delta B_{||}}{B_0}\\ &+\epsilon_\delta\frac{c^2\bar{N}_0\bar{T}}{B_0^2\Omega}\hat{\mathbf{b}}\times\bar{\nabla}\bar{\nabla}_\perp^2\delta\phi + \epsilon_\delta\hat{\mathbf{b}}\times\bar{\nabla}\int c\bar{\mu}\bar{F}_1 d^3\bar{p}.\end{aligned} \tag{28}$$

In the square bracket of Eq. (24), the first term, i.e., the polarization density, arises from the polarization drift, whereas the second term stems from the induced $\mathbf{E}\times\mathbf{B}_0$ drift. The current density (25) contains the $\mathbf{E}\times\mathbf{B}_0$ current $cq\bar{N}_0\hat{\mathbf{b}}\times\bar{\nabla}\delta\phi/B_0$, the current $3cq^2\bar{N}_0\bar{T}\hat{\mathbf{b}}\times\bar{\nabla}\bar{\nabla}_\perp^2\delta\phi/(2B_0^2\Omega)$, and the duple grad-$\delta B_{||}$ current $2c\bar{N}_0\bar{T}\hat{\mathbf{b}}\times\bar{\nabla}\delta B_{||}/B_0^2$.

Finally, the gyrokinetic Maxwell equations are reduced to

$$\begin{aligned}&\bar{\nabla}_\perp\cdot\left[\left(1+\sum_\alpha\frac{\omega_{p\alpha}^2}{\Omega_\alpha^2}\right)\bar{\nabla}_\perp\delta\phi\right] + \sum_\alpha\frac{4\pi q_\alpha\bar{N}_{\alpha 0}}{B_0}\delta B_{||}\\ &\quad= -4\pi\sum_\alpha q_\alpha\int\bar{F}_{\alpha 1}d^3\bar{p}_\alpha,\end{aligned} \tag{29}$$

$$\begin{aligned}&-\bar{\nabla}_\perp\cdot\left[\left(1+\sum_\alpha\beta_{\alpha 0}\right)\bar{\nabla}_\perp\delta\mathbf{A}\right] + \sum_\alpha\frac{\omega_{p\alpha}^2}{c^2}(\delta\mathbf{A}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\\ &\quad-\sum_\alpha\frac{\omega_{p\alpha}^2}{c\Omega_\alpha}\hat{\mathbf{b}}\times\bar{\nabla}\delta\phi - \sum_\alpha\frac{3\omega_{p\alpha}^2\bar{T}_\alpha}{2q_\alpha B_0\Omega_\alpha^2}\hat{\mathbf{b}}\times\bar{\nabla}\bar{\nabla}_\perp^2\delta\phi\\ &\quad=\frac{4\pi}{c}\sum_\alpha\left(\int q_\alpha\frac{\bar{p}_{\alpha||}}{m_\alpha}\bar{F}_{\alpha 1}d^3\bar{p}_\alpha\hat{\mathbf{b}} + \hat{\mathbf{b}}\times\bar{\nabla}\int c\bar{\mu}_\alpha\bar{F}_{\alpha 1}d^3\bar{p}_\alpha\right),\end{aligned} \tag{30}$$

where $\omega_{p\alpha} = 4\pi\bar{N}_{\alpha 0}q_\alpha^2/m_\alpha$ is the particle plasma frequency and $\beta_{\alpha 0} = 8\pi\bar{N}_{\alpha 0}\bar{T}_\alpha/B_0^2$ is the ratio of kinetic to magnetic energy densities.

VIII. DISCUSSION

In this work, a detailed gyrokinetic system is derived via the purely pushforward transformation approach with both retention of the FLR effect and a long wavelength limit in the presence of electromagnetic fluctuations. Compared with the other two approaches, this approach can intuitively reveal the properties of gyrokinetic theory and make the physical mechanism more clear.

With the detailed expression of the first-order gauge scalar field S_1, the systematical analysis on the gyrocenter gyroradius $\boldsymbol{\rho}_y$, which is calculated via the pushforward transformation on the guiding-center gyroradius $\boldsymbol{\rho}_u$, indicates that the motion of the gyrocenter does not contain the entire drifts. In other words, if the gyrocenter motion contains the total drifts in another devised gyrocenter coordinate system, the gyrocenter polarization effect will not exist. Therefore, the key to understanding the gyrokinetic effect is the gyrocenter gyroradius. In addition, the way to decouple the particle motion is closely related to the choice of the gyrokinetic model, which results in the existence of different expressions of gyrokinetic systems. However, the fluctuations ultimately derived via diverse models are coherent with each other.

With these preparations, the moments of distribution are obtained by the purely pushforward transformation approach. The moments have the same form as the results from the purely pullback approach and the conventional approach, except that the variables are different, i.e., $n(\bar{\mathbf{X}})$, $n(\mathbf{x})$ and $n(\mathbf{X})$. This shows that these three approaches are identical not only in principle but also in practice, at least for the linear moments.

From the perspective of polarization and magnetization, the polarization charge density, polarization, and magnetization current density are revealed by the polarization vector and magnetization vector. For the equilibrium flow, the magnetization of the gyrocenter produces a diamagnetic current, and it also provides a current to cancel out the curvature and grad-B_0 drift current. According to the definition of magnetization current, the physical pictures are presented. For the macroscopic perturbed flow, by incorporating the full FLR effect, the polarization current makes no contribution to the macroscopic flow, and the magnetization currents are equivalent to adding an extra FLR effect on the gyrocenter drift current.

In this work, the derivation of pushforward transformation only reaches an order of ϵ_δ. The second-order generating vector field $\mathbf{G}_2$ and the second-order gauge scalar field S_2 are beyond the scope of this work. Thus, the transformed gyrocenter phase-space gyroradius is valid up to the order of $\epsilon_B\epsilon_\delta$, and the gyrocenter moments are valid to the order of ϵ_δ. For the perturbation analysis on the order of ϵ_δ^2, the second-order Hamiltonian including ponderomotive-force-like terms should be brought back, the second-order gyrocenter transformation should be used, and the high order terms in the perturbation analysis on the order of ϵ_δ are needed. In this way, the calculation of fluid moments with a pushforward transformation may be very tedious. Moreover, it might be convenient to derive them by the second-order pullback transformation of the gyrocenter phase-space distribution.

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APPENDIX A: DERIVATION OF MOMENT EQUATIONS

Using the equation

$$
\begin{aligned}
Q\exp\left(\boldsymbol{\rho}_y\cdot\bar{\nabla}\right)\delta &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}QC_m^n\rho_x^n\rho_y^{m-n}\partial_x^n\partial_y^{m-n}\delta \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}\partial_x\left[QC_m^n\rho_x^n\rho_y^{m-n}\partial_x^{n-1}\partial_y^{m-n}\delta\right] + (-1)^1\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}\partial_x\left[QC_m^n\rho_x^n\rho_y^{m-n}\right]\partial_x^{n-1}\partial_y^{m-n}\delta \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}\partial_x\left[QC_m^n\rho_x^n\rho_y^{m-n}\partial_x^{n-1}\partial_y^{m-n}\delta\right] + (-1)^1\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}\partial_x\left[\partial_x\left[QC_m^n\rho_x^n\rho_y^{m-n}\right]\partial_x^{n-2}\partial_y^{m-n}\delta\right] \\
&\quad + \cdots + (-1)^{m-1}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}\partial_y\left[\partial_x^n\partial_y^{m-n-1}\left[QC_m^n\rho_x^n\rho_y^{m-n}\right]\delta\right] \\
&\quad + (-1)^m\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}\partial_x^n\partial_y^{m-n}\left[QC_m^n\rho_x^n\rho_y^{m-n}\right]\delta,
\end{aligned}
$$

where $Q = g(T_\epsilon^{-1}\mathbf{v})\bar{F}B_{||}^*/(m^2)$, Eq. (10) can be obtained. There are two ambiguities during the derivation of moment equations when the full FLR effect is retained, the first of which is the infinite summation in Eq. (10). Since we consider only linear moments and truncate the moments to the order of ϵ, the summation can be converted into the exponential function

$$\Lambda(\mathbf{r}) = \sum_0^\infty (-1)^n \frac{1}{n!}(\bar{\nabla}\cdot)^n \int \bar{F}_0 \overbrace{\boldsymbol{\rho}_u \cdots \boldsymbol{\rho}_u}^{n} g_0 d\bar{p}^3 + \epsilon_\delta \int \exp(-\boldsymbol{\rho}_u\cdot\bar{\nabla})\bar{F}_1 g_0 d\bar{p}^3$$
$$-\epsilon_\delta\bar{\nabla}\cdot\int \bar{F}_0 \exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\boldsymbol{\rho}_y' g_0 d\bar{p}^3 + \epsilon_\delta\int \bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right) g_1 d\bar{p}^3.$$

When setting $g(\mathbf{v})$ equal to 1, the moment equation stands for the particle density

$$n(\mathbf{r}) = \bar{N}_0 + \epsilon_\delta \int \bar{F}_1 J_0 d^3\bar{p} + \epsilon_\delta \bar{\nabla}\cdot\left\langle \bar{F}_0 \exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\frac{q}{B_0\Omega}\left(\delta\tilde{\phi}_u^*\sqrt{\frac{B_0}{2m\bar{\mu}}}\hat{\bar{\boldsymbol{\rho}}} - \sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\theta}}} - \frac{\partial\tilde{\Phi}_u^*}{\partial\bar{p}_{||}}\frac{\mathbf{B}_0^*}{b_{||}^*}\right)\right\rangle_{\bar{p}}.$$

The integral of $\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\tilde{\Phi}_u^*$ (5) will lead to the second ambiguity, but it can be avoided through integral by parts

$$\int \frac{q\bar{F}_0}{B_0}\frac{\partial}{\partial\bar{\theta}}\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}d\bar{\theta} = \int\frac{q\bar{F}_0}{B_0}\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}d\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right) = -\int\frac{q\bar{F}_0}{B_0}\frac{\partial\delta\tilde{\phi}_u^*}{\partial\bar{\mu}}\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)d\bar{\theta}.$$

With

$$\bar{\nabla}\cdot\left\langle \bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\frac{q}{B_0\Omega}\delta\tilde{\phi}_u^*\sqrt{\frac{B_0}{2m\bar{\mu}}}\hat{\bar{\boldsymbol{\rho}}}\right\rangle_{\bar{p}} = -\left\langle\frac{q\bar{F}_0}{B_0}\frac{\partial}{\partial\bar{\mu}}\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\delta\tilde{\phi}_u^*\right\rangle_{\bar{p}},$$

the density is finally derived

$$n(\mathbf{r}) = \bar{N}_0 + \epsilon_\delta\int\bar{F}_1 J_0 d^3\bar{p} - \epsilon_\delta\left\langle\frac{q\bar{F}_0}{B_0}\frac{\partial}{\partial\bar{\mu}}\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\delta\tilde{\phi}_u^*\right\rangle_{\bar{p}} - \epsilon_\delta\left\langle\frac{q\bar{F}_0}{B_0}\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\frac{\partial\delta\tilde{\phi}_u^*}{\partial\bar{\mu}}\right\rangle_{\bar{p}}$$
$$= \bar{N}_0 + \epsilon_\delta\int\bar{F}_1 J_0 d^3\bar{p} + \epsilon_\delta\frac{q\bar{N}_0}{\bar{T}}\delta\phi\left(\langle J_0^2\rangle_{\bar{p}} - 1\right) - \epsilon_\delta\frac{q\bar{N}_0}{c\bar{T}}\left\langle\sqrt{\frac{2\bar{\mu}B_0}{m}}iJ_0J_1\right\rangle_{\bar{p}}\frac{\mathbf{k}_\perp\times\hat{\mathbf{b}}}{k_\perp}\cdot\delta\mathbf{A}_\perp.$$

When setting $g(\mathbf{v})$ equal to $qT_\epsilon^{-1}\mathbf{v}$, the moment equation stands for the current density

$$\mathbf{J} = \epsilon_B\frac{\hat{\mathbf{b}}}{B_0}\times\bar{\nabla}(c\bar{N}_0\bar{T}) + \epsilon_\delta\int q\bar{F}_1\left[\frac{\bar{p}_{||}}{m}J_0\hat{\mathbf{b}} - iJ_1\frac{\mathbf{k}_\perp\times\hat{\mathbf{b}}}{k_\perp}\sqrt{\frac{2\bar{\mu}B_0}{m}}\right]d^3\bar{p}$$
$$+\epsilon_\delta q\bar{\nabla}\cdot\left\langle\bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\frac{q}{B_0\Omega}\left(\delta\tilde{\phi}_u^*\sqrt{\frac{B_0}{2m\bar{\mu}}}\hat{\bar{\boldsymbol{\rho}}} - \sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\theta}}} - \frac{\partial\tilde{\Phi}_u^*}{\partial\bar{p}_{||}}\frac{\mathbf{B}_0^*}{b_{||}^*}\right)\left(\sqrt{\frac{2\bar{\mu}B_0}{m}}\hat{\bar{\boldsymbol{\theta}}} + \frac{\bar{p}_{||}}{m}\frac{\mathbf{b}^*}{b_{||}^*}\right)\right\rangle_{\bar{p}}$$
$$+\epsilon_\delta q\left\langle\bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\left[\frac{\partial\langle q\delta\phi_u^*\rangle}{\partial\bar{p}_{||}}\frac{\mathbf{b}^*}{b_{||}^*} + \frac{c}{B_{||}^*}\hat{\mathbf{b}}\times\bar{\nabla}\langle\delta\phi_u^*\rangle + \sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{q}{B_0}\frac{\partial\langle\delta\phi_u^*\rangle}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\theta}}} - \frac{q}{B_0}\frac{\partial}{\partial\bar{\theta}}\right.\right.$$
$$\left.\left.\times\left(\delta\tilde{\phi}_u^*\sqrt{\frac{B_0}{2m\bar{\mu}}}\hat{\bar{\boldsymbol{\rho}}} - \sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\theta}}} - \frac{\partial\tilde{\Phi}_u^*}{\partial\bar{p}_{||}}\frac{B_0\mathbf{B}_0^*}{B_{||}^*}\right)\right]\right\rangle_{\bar{p}} = \epsilon_B\frac{\hat{\mathbf{b}}}{B_0}\times\bar{\nabla}(c\bar{N}_0\bar{T}) + \epsilon_\delta\int q\bar{F}_1\left[\frac{\bar{p}_{||}}{m}J_0\hat{\mathbf{b}} - iJ_1\frac{\mathbf{k}_\perp\times\hat{\mathbf{b}}}{k_\perp}\sqrt{\frac{2\bar{\mu}B_0}{m}}\right]d^3\bar{p}$$
$$+\epsilon_\delta\bar{\nabla}\cdot\left\langle\frac{q^2\bar{F}_0}{m\Omega}\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\delta\tilde{\phi}_u^*\hat{\bar{\boldsymbol{\rho}}}\hat{\bar{\boldsymbol{\theta}}}\right\rangle_{\bar{p}} + \epsilon_\delta\left\langle\frac{q^2\bar{F}_0}{B_0}\frac{\partial}{\partial\bar{\mu}}\left(\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\delta\tilde{\phi}_u^*\right)\frac{\bar{p}_{||}}{m}\frac{\mathbf{b}^*}{b_{||}^*}\right\rangle_{\bar{p}}$$
$$+\epsilon_\delta\left\langle\frac{q^2\bar{F}_0}{B_0}\left[\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}\sqrt{\frac{2\bar{\mu}B_0}{m}}\hat{\bar{\boldsymbol{\rho}}} - \exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\frac{\partial\delta\tilde{\phi}_u^*}{\partial\bar{\mu}}\sqrt{\frac{2\bar{\mu}B_0}{m}}\hat{\bar{\boldsymbol{\theta}}}\right]\right\rangle_{\bar{p}} + \epsilon_\delta q\left\langle\bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\left[\frac{\partial\langle q\delta\phi_u^*\rangle}{\partial\bar{p}_{||}}\frac{\mathbf{b}^*}{b_{||}^*} + \frac{c}{B_{||}^*}\hat{\mathbf{b}}\right.\right.$$
$$\left.\left.\times\bar{\nabla}\langle\delta\phi_u^*\rangle + \sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{q}{B_0}\frac{\partial\langle\delta\phi_u^*\rangle}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\theta}}} - \frac{q}{B_0}\frac{\partial\delta\tilde{\phi}_u^*}{\partial\bar{\theta}}\sqrt{\frac{B_0}{2m\bar{\mu}}}\hat{\bar{\boldsymbol{\rho}}} - \frac{q}{B_0}\delta\tilde{\phi}_u^*\sqrt{\frac{B_0}{2m\bar{\mu}}}\hat{\bar{\boldsymbol{\theta}}} + \frac{q}{B_0}\sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{\partial\delta\tilde{\phi}_u^*}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\theta}}} - \frac{q}{B_0}\sqrt{\frac{2\bar{\mu}B_0}{m}}\frac{\partial\tilde{\Phi}_u^*}{\partial\bar{\mu}}\hat{\bar{\boldsymbol{\rho}}} + \frac{\partial\delta\tilde{\phi}_u^*}{\partial\bar{p}_{||}}\frac{q\mathbf{B}_0^*}{B_{||}^*}\right]\right\rangle_{\bar{p}}.$$

After some reduction steps, the current density can be obtained,

$$\mathbf{J} = \epsilon_B \frac{\hat{\mathbf{b}}}{B_0} \times \bar{\nabla}(c\bar{N}_0\bar{T}) + \epsilon_\delta \int q\bar{F}_1 \left[\frac{\bar{p}_{||}}{m} J_0 \hat{\mathbf{b}} - iJ_1 \frac{\mathbf{k}_\perp \times \hat{\mathbf{b}}}{k_\perp} \sqrt{\frac{2\bar{\mu}B_0}{m}} \right] d^3\bar{p} - \epsilon_\delta \frac{q^2\bar{N}_0}{cm} \delta A_{||} \langle J_0^2 \rangle_{\bar{p}} \hat{\mathbf{b}}$$
$$+\epsilon_\delta q \left\langle \bar{F}_0 \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \left[\frac{c}{B_{||}^*} \hat{\mathbf{b}} \times \bar{\nabla} \langle \delta\phi_u^* \rangle + \frac{q}{B_0} \frac{\partial}{\partial \bar{\mu}} \left(\sqrt{\frac{2\bar{\mu}B_0}{m}} \langle \delta\phi_u^* \rangle \right) \hat{\bar{\boldsymbol{\theta}}} - \frac{q}{B_0} \sqrt{\frac{B_0}{2m\bar{\mu}}} \frac{\partial}{\partial \bar{\theta}} \left(\delta\phi_u^* \hat{\bar{\boldsymbol{\rho}}} \right) \right] \right\rangle_{\bar{p}}$$
$$+\epsilon_\delta \bar{\nabla} \cdot \left\langle \frac{q^2 \bar{F}_0}{m\Omega} \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \delta\tilde{\phi}_u^* \hat{\bar{\boldsymbol{\rho}}} \hat{\bar{\boldsymbol{\theta}}} \right\rangle_{\bar{p}} = \epsilon_B \frac{\hat{\mathbf{b}}}{B_0} \times \bar{\nabla}(c\bar{N}_0\bar{T}) + \epsilon_\delta \int q\bar{F}_1 \left[\frac{\bar{p}_{||}}{m} J_0 \hat{\mathbf{b}} - iJ_1 \frac{\mathbf{k}_\perp \times \hat{\mathbf{b}}}{k_\perp} \sqrt{\frac{2\bar{\mu}B_0}{m}} \right] d^3\bar{p}$$
$$- \epsilon_\delta \frac{q^2\bar{N}_0}{cm} \delta A_{||} \langle J_0^2 \rangle_{\bar{p}} \hat{\mathbf{b}} + \epsilon_\delta \frac{q^2\bar{N}_0}{\bar{T}} \left\langle \sqrt{\frac{2\bar{\mu}B_0}{m}} iJ_1 \frac{\hat{\mathbf{b}} \times \mathbf{k}_\perp}{k_\perp} \langle \delta\phi_u^* \rangle \right\rangle_{\bar{p}}.$$

When setting $g(\mathbf{v})$ equal to $mT_\epsilon^{-1}\mathbf{v}T_\epsilon^{-1}\mathbf{v}$, the moment equation stands for the pressure tensor

$$\mathbf{P} = \bar{N}_0\bar{T}\mathbf{I} + \epsilon_\delta \int \left[\frac{\bar{p}_{||}^2}{m} J_0 \hat{\mathbf{b}}\hat{\mathbf{b}} + \bar{\mu}B_0(J_0 + J_2)(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \right] \bar{F}_1 d^3\bar{p} + \epsilon_\delta m \bar{\nabla} \cdot \left\langle \bar{F}_0 \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \frac{q}{B_0\Omega} \right.$$
$$\times \left. \left(\delta\tilde{\phi}_u^* \sqrt{\frac{B_0}{2m\bar{\mu}}} \hat{\bar{\boldsymbol{\rho}}} - \sqrt{\frac{2\bar{\mu}B_0}{m}} \frac{\partial \tilde{\Phi}_u^*}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} - \frac{\partial \tilde{\Phi}_u^*}{\partial \bar{p}_{||}} \frac{\mathbf{B}_0^*}{b_{||}^*} \right) \left(\frac{2\bar{\mu}B_0}{m} \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\theta}}} + \frac{\bar{p}_{||}^2}{m^2} \hat{\mathbf{b}}\hat{\mathbf{b}} \right) \right\rangle_{\bar{p}} + \epsilon_\delta m \left\langle \bar{F}_0 \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \right.$$
$$\times \left\{ \left[\frac{\partial \langle q\delta\phi_u^* \rangle}{\partial \bar{p}_{||}} \frac{\mathbf{b}^*}{b_{||}^*} + \frac{c}{B_{||}^*} \hat{\mathbf{b}} \times \bar{\nabla} \langle \delta\phi_u^* \rangle + \sqrt{\frac{2\bar{\mu}B_0}{m}} \frac{q}{B_0} \frac{\partial \langle \delta\phi_u^* \rangle}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} - \frac{q}{B_0} \frac{\partial}{\partial \bar{\theta}} \left(\delta\tilde{\phi}_u^* \sqrt{\frac{B_0}{2m\bar{\mu}}} \hat{\bar{\boldsymbol{\rho}}} - \sqrt{\frac{2\bar{\mu}B_0}{m}} \frac{\partial \tilde{\Phi}_u^*}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} - \frac{\partial \tilde{\Phi}_u^*}{\partial \bar{p}_{||}} \frac{B_0 \mathbf{B}_0^*}{B_{||}^*} \right) \right] \right.$$
$$\times \left(\sqrt{\frac{2\bar{\mu}B_0}{m}} \hat{\bar{\boldsymbol{\theta}}} + \frac{\bar{p}_{||}}{m} \hat{\mathbf{b}} \right) + \left(\sqrt{\frac{2\bar{\mu}B_0}{m}} \hat{\bar{\boldsymbol{\theta}}} + \frac{\bar{p}_{||}}{m} \hat{\mathbf{b}} \right) \left[\frac{\partial \langle q\delta\phi_u^* \rangle}{\partial \bar{p}_{||}} \frac{\mathbf{b}^*}{b_{||}^*} + \frac{c}{B_{||}^*} \hat{\mathbf{b}} \times \bar{\nabla} \langle \delta\phi_u^* \rangle + \sqrt{\frac{2\bar{\mu}B_0}{m}} \frac{q}{B_0} \frac{\partial \langle \delta\phi_u^* \rangle}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} - \frac{q}{B_0} \frac{\partial}{\partial \bar{\theta}} \right.$$
$$\times \left. \left. \left. \left(\delta\tilde{\phi}_u^* \sqrt{\frac{B_0}{2m\bar{\mu}}} \hat{\bar{\boldsymbol{\rho}}} - \sqrt{\frac{2\bar{\mu}B_0}{m}} \frac{\partial \tilde{\Phi}_u^*}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} - \frac{\partial \tilde{\Phi}_u^*}{\partial \bar{p}_{||}} \frac{B_0 \mathbf{B}_0^*}{B_{||}^*} \right) \right] \right\} \right\rangle_{\bar{p}}.$$

With the equation

$$\langle \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\theta}} \exp\left(-i\mathbf{k}_\perp \cdot \boldsymbol{\rho}_u\right) \rangle = \left[\frac{J_2 \sin 2\alpha}{2} \left(\hat{\mathbf{e}}_y \hat{\mathbf{e}}_x + \hat{\mathbf{e}}_x \hat{\mathbf{e}}_y \right) - J_2 \sin^2\alpha \hat{\mathbf{e}}_x \hat{\mathbf{e}}_x - J_2 \cos^2\alpha \hat{\mathbf{e}}_y \hat{\mathbf{e}}_y + \frac{J_0 + J_2}{2} \left(\hat{\mathbf{e}}_x \hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y \hat{\mathbf{e}}_y \right) \right],$$

the pressure tensor can be obtained,

$$\mathbf{P} = \bar{N}_0\bar{T}\mathbf{I} + \epsilon_\delta \int \left[\frac{\bar{p}_{||}^2}{m} J_0 \hat{\mathbf{b}}\hat{\mathbf{b}} + \bar{\mu}B_0(J_0 + J_2)(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \right] \bar{F}_1 d^3\bar{p} + \bar{\nabla} \cdot \left\langle 2\bar{\mu} \frac{q\bar{F}_0}{\Omega} \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \delta\tilde{\phi}_u^* \sqrt{\frac{B_0}{2m\bar{\mu}}} \hat{\bar{\boldsymbol{\rho}}} \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\theta}}} \right\rangle_{\bar{p}}$$
$$- m \left\langle \frac{2\bar{\mu}B_0}{m} \frac{q\bar{F}_0}{B_0} \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \frac{\partial \delta\tilde{\phi}_u^*}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\theta}}} \right\rangle_{\bar{p}} + m \left\langle \bar{F}_0 \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \left\{ \left[\sqrt{\frac{2\bar{\mu}B_0}{m}} \hat{\bar{\boldsymbol{\theta}}} \frac{c}{B_{||}^*} \hat{\mathbf{b}} \times \bar{\nabla} \langle \delta\phi_u^* \rangle + \frac{c}{B_{||}^*} \hat{\mathbf{b}} \times \bar{\nabla} \langle \delta\phi_u^* \rangle \sqrt{\frac{2\bar{\mu}B_0}{m}} \hat{\bar{\boldsymbol{\theta}}} \right. \right. \right.$$
$$\left. \left. \left. + 2 \frac{2\bar{\mu}B_0}{m} \frac{q}{B_0} \frac{\partial \langle \delta\phi_u^* \rangle}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\theta}}} - 2 \frac{q}{m} \delta\tilde{\phi}_u^* \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\theta}}} - \frac{q}{m} \frac{\partial \delta\phi_u^*}{\partial \bar{\theta}} \left(\hat{\bar{\boldsymbol{\rho}}} \hat{\bar{\boldsymbol{\theta}}} + \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\rho}}} \right) + 2 \frac{q}{B_0} \frac{2\bar{\mu}B_0}{m} \frac{\partial \delta\tilde{\phi}_u^*}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\theta}}} \right] \right\} \right\rangle_{\bar{p}}$$
$$= \bar{N}_0\bar{T}\mathbf{I} + \epsilon_\delta \int \left[\frac{\bar{p}_{||}^2}{m} J_0 \hat{\mathbf{b}}\hat{\mathbf{b}} + \bar{\mu}B_0(J_0 + J_2)(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \right] \bar{F}_1 d^3\bar{p} - m \left\langle \frac{q\bar{F}_0}{B_0} \frac{\partial}{\partial \bar{\mu}} \left[\frac{2\bar{\mu}B_0}{m} \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \delta\tilde{\phi}_u^* \right] \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\theta}}} \right\rangle_{\bar{p}} + m \left\langle \bar{F}_0 \exp\left(-\boldsymbol{\rho}_u \cdot \bar{\nabla}\right) \right.$$
$$\times \left. \left[\sqrt{\frac{2\bar{\mu}B_0}{m}} \hat{\bar{\boldsymbol{\theta}}} \frac{c}{B_{||}^*} \hat{\mathbf{b}} \times \bar{\nabla} \langle \delta\phi_u^* \rangle + \frac{c}{B_{||}^*} \hat{\mathbf{b}} \times \bar{\nabla} \langle \delta\phi_u^* \rangle \sqrt{\frac{2\bar{\mu}B_0}{m}} \hat{\bar{\boldsymbol{\theta}}} + 2 \frac{2\bar{\mu}B_0}{m} \frac{q}{B_0} \frac{\partial \delta\phi_u^*}{\partial \bar{\mu}} \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\theta}}} - \frac{q}{m} \frac{\partial \delta\phi_u^*}{\partial \bar{\theta}} \left(\hat{\bar{\boldsymbol{\rho}}} \hat{\bar{\boldsymbol{\theta}}} + \hat{\bar{\boldsymbol{\theta}}} \hat{\bar{\boldsymbol{\rho}}} \right) \right] \right\rangle_{\bar{p}}$$
$$= \bar{N}_0\bar{T}\mathbf{I} + \epsilon_\delta \int \left[\frac{\bar{p}_{||}^2}{m} J_0 \hat{\mathbf{b}}\hat{\mathbf{b}} + \bar{\mu}B_0(J_0 + J_2)(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \right] \bar{F}_1 d^3\bar{p} + \epsilon_\delta \frac{qB_0\bar{N}_0}{\bar{T}} \langle \bar{\mu} \left[(J_0 + J_2) \langle \delta\phi_u^* \rangle - \delta\phi \right] (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \rangle_{\bar{p}},$$

where the off-diagonal components related to $\hat{\mathbf{b}}$ and terms whose divergence is a higher-order contribution are neglected.

APPENDIX B: DERIVATION OF POLARIZATION AND MAGNETIZATION

The polarization and magnetization currents can be decoupled from the total current with the help of the Vlasov equation and the gyrocenter Liouville theorem. If the polarization current is introduced

$$
\begin{aligned}
\frac{\partial \mathbf{P}_\epsilon}{\partial t} =& -q\sum_1^\infty (-1)^n \frac{1}{n!}(\bar\nabla\cdot)^{n-1}\int \frac{\partial \bar F}{\partial t}\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}\,d\bar P^3 - q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^{n-1}\int \bar F\partial\frac{\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_u}^{n}}{\partial t}\bar{\mathcal{J}}\,d\bar P^3 \\
& -q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^{n-1}\int\frac{\partial}{\partial\bar{\mathbf{X}}}\cdot\left[\dot{\bar{\mathbf{X}}}\bar F\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}\right]d\bar P^3 + q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^{n-1}\int\frac{\partial}{\partial\bar{\mathbf{X}}}\cdot\left[\dot{\bar{\mathbf{X}}}\bar F\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}\right]d\bar P^3 \\
& -q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^{n-1}\int\frac{\partial}{\partial\bar\theta}\left[\dot{\bar\theta}\bar F\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}\right]d\bar P^3 - q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^{n-1}\int\frac{\partial}{\partial\bar p_{||}}\left[\dot{\bar p}_{||}\bar F\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}\right]d\bar P^3 \\
=& \, q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^n\int\dot{\bar{\mathbf{X}}}\bar F\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}\,d\bar P^3 - q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^{n-1}\int\bar F\frac{d\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}}{dt}\bar{\mathcal{J}}\,d\bar P^3 - q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^{n-1} \\
& \times\int\bar F\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\left[\frac{\partial}{\partial\bar p_{||}}\left(\dot{\bar p}_{||}\bar{\mathcal{J}}\right)+\frac{\partial}{\partial\bar\theta}\left(\dot{\bar\theta}\,\bar{\mathcal{J}}\right)+\frac{\partial}{\partial\bar{\mathbf{X}}}\left(\dot{\bar{\mathbf{X}}}\bar{\mathcal{J}}\right)\right]d\bar P^3 - q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^{n-1}\int\frac{d\bar F}{dt}\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}\,d\bar P^3 \\
=& \, q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^n\int\dot{\bar{\mathbf{X}}}\bar F\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}\,d\bar P^3 + q\sum_0^\infty(-1)^n\frac{1}{(n+1)!}(\bar\nabla\cdot)^n\int\bar F\frac{d\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n+1}}{dt}\bar{\mathcal{J}}\,d\bar P^3,
\end{aligned}
$$

where $\bar{\mathcal{J}} = B^*_{||}/(m^2)$ and $d\bar P^3 = d\bar\mu d\bar p_{||}d\bar\theta$, then the current density becomes

$$
\begin{aligned}
\mathbf{J} =& \, q\int\bar F\dot{\bar{\mathbf{X}}}\bar{\mathcal{J}}\,d\bar P^3 + \frac{\partial\mathbf{P}_\epsilon}{\partial t} + q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^n\int\bar F\left(\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\dot{\bar{\mathbf{X}}}-\dot{\bar{\mathbf{X}}}\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\right)\bar{\mathcal{J}}\,d\bar P^3 \\
& + q\sum_0^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^n\int\bar F\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\dot{\boldsymbol{\rho}}_y\bar{\mathcal{J}}\,d\bar P^3 - q\sum_0^\infty(-1)^n\frac{1}{(n+1)!}(\bar\nabla\cdot)^n\int\bar F\frac{d\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n+1}}{dt}\bar{\mathcal{J}}\,d\bar P^3,
\end{aligned}
$$

where

$$
\begin{aligned}
& q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^n\int\bar F\left(\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\dot{\bar{\mathbf{X}}}-\dot{\bar{\mathbf{X}}}\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\right)\bar{\mathcal{J}}\,d\bar P^3 \\
&= -\epsilon_B q\bar\nabla\cdot\int\bar F_0\left(\boldsymbol{\rho}_u\dot{\bar{\mathbf{X}}}_0-\dot{\bar{\mathbf{X}}}_0\boldsymbol{\rho}_u\right)\bar{\mathcal{J}}\,d\bar P^3 + \epsilon_\delta q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^n\int\bar F_0\left(\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\dot{\bar{\mathbf{X}}}-\dot{\bar{\mathbf{X}}}\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\right)\bar{\mathcal{J}}\,d\bar P^3 \\
&\quad + \epsilon_\delta q\sum_1^\infty(-1)^n\frac{1}{n!}(\bar\nabla\cdot)^n\int\bar F_1\left(\overbrace{\boldsymbol{\rho}_u\cdots\boldsymbol{\rho}_u}^{n}\dot{\bar{\mathbf{X}}}_0-\dot{\bar{\mathbf{X}}}_0\overbrace{\boldsymbol{\rho}_u\cdots\boldsymbol{\rho}_u}^{n}\right)\bar{\mathcal{J}}\,d\bar P^3 \\
&= \epsilon_B q\bar\nabla\times\int\bar F_0\boldsymbol{\rho}_u\times\dot{\bar{\mathbf{X}}}_0\bar{\mathcal{J}}\,d\bar P^3 - \epsilon_\delta q\sum_1^\infty\frac{1}{n!}\bar\nabla\cdot\int\bar F_0\left(-\boldsymbol{\rho}_u\cdot\bar\nabla\right)^{n-1}\left(\boldsymbol{\rho}_u\dot{\bar{\mathbf{X}}}_1-\dot{\bar{\mathbf{X}}}_1\boldsymbol{\rho}_u\right)\bar{\mathcal{J}}\,d\bar P^3 \\
&\quad - \epsilon_\delta q\sum_1^\infty\frac{1}{n!}\bar\nabla\cdot\int\bar F_0 n\left(-\boldsymbol{\rho}_u\cdot\bar\nabla\right)^{n-1}\left(\boldsymbol{\rho}'_y\dot{\bar{\mathbf{X}}}_0-\dot{\bar{\mathbf{X}}}_0\boldsymbol{\rho}'_y\right)\bar{\mathcal{J}}\,d\bar P^3 - \epsilon_\delta q\sum_1^\infty\frac{1}{n!}\bar\nabla\cdot\int\left(-\boldsymbol{\rho}_u\cdot\bar\nabla\right)^{n-1}\bar F_1\left(\boldsymbol{\rho}_u\dot{\bar{\mathbf{X}}}_0-\dot{\bar{\mathbf{X}}}_0\boldsymbol{\rho}_u\right)\bar{\mathcal{J}}\,d\bar P^3 \\
&= -\epsilon_\delta q\sum_1^\infty\frac{1}{n!}\bar\nabla\times\int\left(-\boldsymbol{\rho}_u\cdot\bar\nabla\right)^{n-1}\left[\bar F_0\left(\dot{\bar{\mathbf{X}}}_1\times\boldsymbol{\rho}_u\right)\right]\bar{\mathcal{J}}\,d\bar P^3 - \epsilon_\delta q\sum_1^\infty\frac{1}{(n-1)!}\bar\nabla\times\int\bar F_0\left(-\boldsymbol{\rho}_u\cdot\bar\nabla\right)^{n-1}\left(\dot{\bar{\mathbf{X}}}_0\times\boldsymbol{\rho}'_y\right)\bar{\mathcal{J}}\,d\bar P^3 \\
&\quad - \epsilon_\delta q\sum_1^\infty\frac{1}{n!}\bar\nabla\times\int\left(-\boldsymbol{\rho}_u\cdot\bar\nabla\right)^{n-1}\left[\bar F_1\left(\dot{\bar{\mathbf{X}}}_0\times\boldsymbol{\rho}_u\right)\right]\bar{\mathcal{J}}\,d\bar P^3
\end{aligned}
$$

and

$$
\begin{aligned}
&q\sum_{0}^{\infty}(-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\int\bar{F}\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\dot{\boldsymbol{\rho}}_y\bar{\mathcal{J}}d\bar{P}^3-q\sum_{0}^{\infty}(-1)^n\frac{1}{(n+1)!}(\bar{\nabla}\cdot)^n\int\bar{F}\frac{d\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n+1}}{dt}\bar{\mathcal{J}}d\bar{P}^3\\
&\quad=-\epsilon_B\frac{q}{2}\bar{\nabla}\cdot\int\left\{\bar{F}_0\left[\boldsymbol{\rho}_u(\dot{\boldsymbol{\rho}}_y)_0-(\dot{\boldsymbol{\rho}}_y)_0\boldsymbol{\rho}_u\right]\right\}\bar{\mathcal{J}}d\bar{P}^3-\epsilon_\delta q\sum_{1}^{\infty}\frac{n}{(n+1)!}\bar{\nabla}\cdot\int\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)^{n-1}\bar{F}_1\left(\boldsymbol{\rho}_u(\dot{\boldsymbol{\rho}}_y)_0-(\dot{\boldsymbol{\rho}}_y)_0\boldsymbol{\rho}_u\right)\bar{\mathcal{J}}d\bar{P}^3\\
&\qquad-\epsilon_\delta q\sum_{1}^{\infty}\frac{n}{(n+1)!}\bar{\nabla}\cdot\int\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)^{n-1}\bar{F}_0\left(\boldsymbol{\rho}_u(\dot{\boldsymbol{\rho}}_y)_1-(\dot{\boldsymbol{\rho}}_y)_1\boldsymbol{\rho}_u\right)\bar{\mathcal{J}}d\bar{P}^3\\
&\qquad-\epsilon_\delta q\sum_{1}^{\infty}\frac{n^2}{(n+1)!}\bar{\nabla}\cdot\int\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)^{n-1}\bar{F}_0\left[\boldsymbol{\rho}'_y(\dot{\boldsymbol{\rho}}_y)_0-(\dot{\boldsymbol{\rho}}_y)_0\boldsymbol{\rho}'_y\right]\bar{\mathcal{J}}d\bar{P}^3\\
&\qquad-\epsilon_\delta q\sum_{2}^{\infty}\frac{n^2-n}{(n+1)!}(\bar{\nabla}\cdot)^2\int\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)^{n-2}\bar{F}_0\left[(\dot{\boldsymbol{\rho}}_y)_0\boldsymbol{\rho}'_y\boldsymbol{\rho}_u-(\dot{\boldsymbol{\rho}}_y)_0\boldsymbol{\rho}_u\boldsymbol{\rho}'_y\right]\bar{\mathcal{J}}d\bar{P}^3\\
&\quad=\epsilon_B\frac{q}{2}\bar{\nabla}\times\int\bar{F}_0\boldsymbol{\rho}_u\times(\dot{\boldsymbol{\rho}}_y)_0\bar{\mathcal{J}}d\bar{P}^3-\epsilon_\delta q\sum_{1}^{\infty}\frac{n}{(n+1)!}\bar{\nabla}\times\int\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)^{n-1}\bar{F}_1(\dot{\boldsymbol{\rho}}_y)_0\times\boldsymbol{\rho}_u\bar{\mathcal{J}}d\bar{P}^3\\
&\qquad-\epsilon_\delta q\sum_{1}^{\infty}\frac{n}{(n+1)!}\bar{\nabla}\times\int\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)^{n-1}\bar{F}_0(\dot{\boldsymbol{\rho}}_y)_1\times\boldsymbol{\rho}_u\bar{\mathcal{J}}d\bar{P}^3\\
&\qquad-\epsilon_\delta q\sum_{1}^{\infty}\frac{n^2}{(n+1)!}\bar{\nabla}\times\int\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)^{n-1}\bar{F}_0\left[(\dot{\boldsymbol{\rho}}_y)_0\times\boldsymbol{\rho}'_y\right]\bar{\mathcal{J}}d\bar{P}^3\\
&\qquad-\epsilon_\delta q\sum_{2}^{\infty}\frac{n^2-n}{(n+1)!}\bar{\nabla}\times\int\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)^{n-2}\bar{F}_0\left\{\boldsymbol{\rho}_u\times\left[(\dot{\boldsymbol{\rho}}_y)_0\cdot\bar{\nabla}\right]\boldsymbol{\rho}'_y\right\}\bar{\mathcal{J}}d\bar{P}^3.
\end{aligned}
$$

In this way, the current can be rewritten in the form

$$\mathbf{J}=\bar{\mathbf{J}}_y+\frac{\partial\mathbf{P}_\epsilon}{\partial t}+c\bar{\nabla}\times\mathbf{M}_\epsilon.$$

The direct solution of $\mathbf{M}_\epsilon$ will encounter the tedious infinite summation problem. Since the dominant term of $\left(d\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n+1}/dt\right)$ is

$$\frac{d\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n+1}}{dt}=\frac{\Omega}{B_0}\frac{\partial\,\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n+1}}{\partial\bar{\theta}}\frac{\partial\bar{\mathcal{H}}_y}{\partial\bar{\mu}}+\cdots,$$

with the equations

$$
\begin{aligned}
q\sum_{1}^{\infty}(-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\int\dot{\bar{\mathbf{X}}}\bar{F}\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\bar{\mathcal{J}}d\bar{P}^3&=\epsilon_\delta q\sum_{1}^{\infty}(-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\int\dot{\bar{\mathbf{X}}}_1\bar{F}_0\overbrace{\boldsymbol{\rho}_u\cdots\boldsymbol{\rho}_u}^{n}\bar{\mathcal{J}}d\bar{P}^3+\epsilon_\delta q\sum_{1}^{\infty}(-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\\
&\quad\times\int\dot{\bar{\mathbf{X}}}_0\bar{F}_1\overbrace{\boldsymbol{\rho}_u\cdots\boldsymbol{\rho}_u}^{n}\bar{\mathcal{J}}d\bar{P}^3+\epsilon_\delta q\sum_{1}^{\infty}(-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\int\dot{\bar{\mathbf{X}}}_0\bar{F}_0\left(\overbrace{\boldsymbol{\rho}_y\cdots\boldsymbol{\rho}_y}^{n}\right)_1\bar{\mathcal{J}}d\bar{P}^3\\
&=\epsilon_\delta q\sum_{1}^{\infty}(-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\int\left[\epsilon_\delta q\frac{\partial\langle\delta\phi_u^*\rangle}{\partial\bar{p}_{||}}\hat{\mathbf{b}}+\epsilon_\delta\frac{c}{B_{||}^*}\hat{\mathbf{b}}\times\bar{\nabla}\langle\delta\phi_u^*\rangle\right]\bar{F}_0\overbrace{\boldsymbol{\rho}_u\cdots\boldsymbol{\rho}_u}^{n}\bar{\mathcal{J}}d\bar{P}^3+\epsilon^2
\end{aligned}
$$

and

$$\bar{\nabla}\cdot\left[\epsilon_\delta q\frac{\partial\langle\delta\phi_u^*\rangle}{\partial\bar{p}_{||}}\hat{\mathbf{b}}+\epsilon_\delta\frac{c}{B_{||}^*}\hat{\mathbf{b}}\times\bar{\nabla}\langle\delta\phi_u^*\rangle\right]\sim\epsilon_\delta\epsilon_{||},$$

the polarization current clearly provides a higher-order contribution. With this conclusion,

$$q\sum_{0}^{\infty}(-1)^n\frac{1}{(n+1)!}(\bar{\nabla}\cdot)^n\int\bar{F}\frac{d\boldsymbol{\rho}_y^{n+1}}{dt}\bar{\mathcal{J}}d\bar{P}^3$$

can be artificially added into the magnetization current, and the magnetization current can be derived as

$$
\begin{aligned}
\mathbf{J}_m(\bar{\mathbf{X}}) = c\bar{\nabla}\times\mathbf{M}_\epsilon = \epsilon_\delta q\int\left[\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)-1\right]\bar{F}_1\dot{\bar{\mathbf{X}}}_0\bar{\mathcal{J}}\,d\bar{P}^3 + \epsilon_\delta q\int\bar{F}_0\left[\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)-1\right]\dot{\bar{\mathbf{X}}}_1\bar{\mathcal{J}}\,d\bar{P}^3 \\
+ q\sum_0^1(-1)^n\frac{1}{n!}(\bar{\nabla}\cdot)^n\int\bar{F}_0\overbrace{\boldsymbol{\rho}_u\cdots\boldsymbol{\rho}_u}^{n}\left(\dot{\boldsymbol{\rho}}_y\right)_0\bar{\mathcal{J}}\,d\bar{P}^3 + \epsilon_\delta q\int\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\bar{F}_1\left(\dot{\boldsymbol{\rho}}_y\right)_0\bar{\mathcal{J}}\,d\bar{P}^3 - \epsilon_\delta q\bar{\nabla} \\
\times\int\bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\boldsymbol{\rho}_y'\left[\dot{\bar{\mathbf{X}}}_0+\left(\dot{\boldsymbol{\rho}}_y\right)_0\right]\bar{\mathcal{J}}\,d\bar{P}^3 + \epsilon_\delta q\int\bar{F}_0\exp\left(-\boldsymbol{\rho}_u\cdot\bar{\nabla}\right)\left(\dot{\boldsymbol{\rho}}_y\right)_1\bar{\mathcal{J}}\,d\bar{P}^3 \\
= -\epsilon_B\bar{\nabla}\times\left(\frac{c\bar{N}_0\bar{T}}{B_0}\hat{\mathbf{b}}\right) - \epsilon_\delta\frac{q^2\bar{N}_0}{cm}\delta A_{\|}\langle J_0(J_0-1)\rangle_{\bar{p}}\hat{\mathbf{b}} - \epsilon_\delta\frac{cq\bar{N}_0}{B_{\|}^*}\hat{\mathbf{b}}\times\bar{\nabla}\langle\langle\delta\phi_u^*\rangle\rangle_{\bar{p}} \\
+ \epsilon_\delta\frac{q^2\bar{N}_0}{\bar{T}}\left\langle\sqrt{\frac{2\bar{\mu}B_0}{m}}iJ_1\frac{\hat{\mathbf{b}}\times\mathbf{k}_\perp}{k_\perp}\langle\delta\phi_u^*\rangle\right\rangle_{\bar{p}} + \epsilon_\delta\int q\bar{F}_1\left[\frac{\bar{p}_{\|}}{m}(J_0-1)\hat{\mathbf{b}} - iJ_1\frac{\mathbf{k}_\perp\times\hat{\mathbf{b}}}{k_\perp}\sqrt{\frac{2\bar{\mu}B_0}{m}}\right]\bar{\mathcal{J}}\,d^3\bar{P}.
\end{aligned}
$$

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