Shearing rate of time-dependent $E \times B$ flow


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Theory of $E \times B$ shear suppression of turbulence in toroidal geometry [Phys. Plasmas 2, 1648 (1995)] is extended to include fast time variations of the $E \times B$ flows often observed in nonlinear simulations of tokamak turbulence. It is shown that the quickly time varying components of the $E \times B$ flows, while they typically contribute significantly to the instantaneous $E \times B$ shearing rate, are less effective than the slowly time varying components in suppressing turbulence. This is because the shear flow pattern changes before eddies get distorted enough. The effective $E \times B$ shearing rate capturing this important physics is analytically derived and estimated from zonal flow statistics of gyrofluid simulation. This provides new insights into understanding recent gyrofluid and gyrokinetic simulations that yield a reduced, but not completely quenched, level of turbulence in the presence of turbulence-driven zonal flows. © 1999 American Institute of Physics.

I. INTRODUCTION

There is accumulating evidence that $E \times B$ shear suppression of turbulence is the most likely mechanism to be responsible for various forms of confinement enhancement.\(^1\)–\(^16\) The theory of $E \times B$ shear suppression was first developed in cylindrical geometry.\(^17\) It is valid when the time variation of $E_r$ is much slower than the eddy turn-over time. Extension to a shaped tokamak geometry\(^18\) has been useful for comparisons to experimental data,\(^1\)–\(^3\),\(^5\)–\(^8\),\(^9\)–\(^11\),\(^13\)–\(^14\) which indeed show such a relatively slow time variation of the macroscopic $E_r$.

On the other hand, early flux-tube gyrofluid simulations of ITG turbulence\(^19\)–\(^22\) have observed relatively small radial scale (several ion gyro-radius) fluctuating sheared $E \times B$ flows driven by turbulence. Flows can be generated by the Reynolds’ stress\(^23\) and can be considered as a nonlinear instability associated with inverse cascade of the turbulent spectra.\(^24\) Recently, the importance of these small scale zonal flows in regulating turbulence in tokamaks has emerged as a serious possibility. It has appeared consistently in the nonlinear gyrokinetic and gyrofluid simulations of Ion Temperature Gradient (ITG) turbulence as various flux-tube codes were developed independently.\(^25\),\(^26\) Moreover, the recent gyrokinetic simulations\(^27\) in both full torus and annulus geometry with various boundary conditions have produced results that exhibit the importance of the fluctuating flows with similar characteristics as those in the flux-tube simulations,\(^22\),\(^25\) when radial variations of the pressure gradient are mild.

We note that early global gyrokinetic simulations had a relatively small system size in ion gyro-radius units, and, consequently, had rather sharp radial variations of pressure gradient. They either did not include\(^28\),\(^29\) or did not observe\(^30\) the small scale zonal flows. As computing power became sufficient to handle larger system size, the finer scale flows began to appear in global gyrokinetic simulations,\(^31\) although its effect on steady state transport was not observed to be as significant as that seen in the flux tube simulations.

These flows observed in simulations contain significant components with radial scales and frequencies comparable to those of the turbulence. It is therefore of vital importance to extend the nonlinear theory of turbulence decorrelation by $E \times B$ shear\(^17\),\(^18\),\(^32\) to address the role of fast time varying $E \times B$ flow shear in regulating turbulence. This will lead to a better quantitative understanding of the nonlinear simulation results.

In this paper, it is shown that the quickly time varying components, while they typically contribute significantly to the instantaneous $E \times B$ shearing rate, are less effective than the slowly time varying components in suppressing turbulence, since the flow shear pattern changes before the eddies get distorted enough. Effective $E \times B$ shearing rate, $\omega_{\text{Eff}}$ capturing this important physics is analytically derived. We predict a significant reduction in the radial correlation length of turbulence when $\omega_{\text{Eff}}$ becomes comparable to the decorrelation rate of the ambient turbulence. Indeed, an estimated value from zonal flow statistics of gyrofluid simulation is comparable to the maximum linear growth rate. This is in qualitative agreement with the broadening of $k_r$ spectrum of turbulence observed in the recent gyrokinetic simulations.\(^27\) This could also provide new insights into understanding simulations that observe a reduced, not completely quenched level of turbulence.\(^33\),\(^27\) The instantaneous $E \times B$ shearing rate observed in these simulations is much higher than the maximum linear growth rate, and is an overestimate of the shearing effect. Our results also suggest the relative importance of the low frequency part of the zonal flow.

II. TURBULENCE DECORRELATION IN THE PRESENCE OF TIME-DEPENDENT $E \times B$ SHEAR

In this section, decorrelation of fluctuations is estimated via a two point nonlinear analysis in the presence of this time varying $E \times B$ flow shear. Following the previous work,\(^17\),\(^18\),\(^32\) we start from a one-field fluid model in which the fluctuating field $\delta H$ is convected by the fluctuating zonal flows $V_F$, and the fluctuating $E \times B$ flow $\vec{V}_F$ associated with the ambient turbulence excluding the zonal flows.
\[
(\partial/\partial t + V_E \cdot \nabla + \tilde{V}_E \cdot \nabla) \delta H = S,
\]
where \(V_E = B \times \nabla \Phi / B^2\), \(\tilde{V}_E = B \times \nabla \delta \Phi / B^2\), and \(S\) is the driving source of the turbulence. Linear dissipation and sub-dominant nonlinearities other than \(E \times B\) nonlinearity are ignored for simplicity. We use the coordinate system in which \(B = \nabla \phi \times \nabla \psi + \text{I} (\psi) \nabla \psi\), \(B_\phi = \text{I} (\psi) / R\), and \(B_\psi = |\nabla \psi| / R\). Here, \(\phi\) is the toroidal angle and \(\psi\) is the poloidal flux. We consider a model problem in which \(\Phi\) is a flux function, but depends on time in the following way:

\[
\Phi(\psi, t) = \Phi_0(\psi) \exp[-i \omega_f (t - t_0)],
\]
where \(t_0\) specifies the initial phase. The two-point correlation evolution equation is then derived following the standard procedure of symmetrization with respect to \((\psi_1, \phi_1, \theta_1)\) and \((\psi_2, \phi_2, \theta_2)\) followed by an ensemble average. Here, \(\theta\) is the angle-like coordinate along the field

\[
\left[\frac{\partial}{\partial t} + \psi_\phi \exp[-i \omega_f (t - t_0)] \frac{\partial}{\partial \phi_\phi} - \mathcal{D}_{\phi}^{\text{eff}} \frac{\partial^2}{\partial \phi_\phi^2}\right] \times \langle \delta H(1) \delta H(2) \rangle = S_2.
\]

The corresponding radial shear of the time-dependent angular frequency is given by

\[
\Omega_E(\psi, t) = -\frac{\partial^2}{\partial \phi^2} \Phi_0(\psi) \exp[-i \omega_f (t - t_0)]
= \Omega_\phi \exp[-i \omega_f (t - t_0)].
\]

In Eq. (2), \(S_2\) is the source term for the two-point correlation function and the \(E \times B\) nonlinearity is approximated as a turbulent diffusion along the perpendicular direction. At small separation, the relative diffusion \(\mathcal{D}_{\phi}^{\text{eff}}\) has the following asymptotic form:

\[
\mathcal{D}_{\phi}^{\text{eff}} = 2 \mathcal{D}_{\phi}^{\text{eff}} \left[ \frac{\psi_\phi}{\Delta \psi_0} + \frac{\theta_\phi^2}{\Delta \theta} + \frac{\phi_\phi^2}{\Delta \phi} \right],
\]

where \(\Delta r_0 = \Delta \psi_0 / R B_\phi\) and \(\Delta \theta, \Delta \phi\) are the correlation lengths in the radial and toroidal directions, respectively. \(\mathcal{D}_{\phi}^{\text{eff}} = \Delta \omega_f / 4\) is proportional to the diffusion coefficient at large separation.

The decorrelation dynamics due to the coupling of the flow shear and turbulent diffusion can be studied by taking various moments of the left hand side (LHS) of Eq. (2),

\[
\begin{align*}
\partial_t \langle \psi_\phi^2 \rangle &= 0, \\
\partial_t \langle \theta_\phi^2 \rangle &= 0, \\
\partial_t \langle \phi_\phi^2 \rangle &= 4 \mathcal{D}_{\phi}^{\text{eff}} \left[ \frac{\theta_\phi^2}{\Delta \eta^2} + \frac{\phi_\phi^2}{\Delta \phi^2} \right] + 2 \Omega_\phi \exp[-i \omega_f (t - t_0)] \langle \psi_\phi \phi_\phi \rangle.
\end{align*}
\]

Here,

\[
\langle A(\theta_\phi, \phi_\phi, \psi_\phi) \rangle = \int d\theta_\phi' d\phi_\phi' d\psi_\phi' G(\theta_\phi, \phi_\phi, \psi_\phi | \theta_\phi', \phi_\phi', \psi_\phi') A(\theta_\phi, \phi_\phi, \psi_\phi),
\]

and \(G\) is the two point Green’s function for the LHS of Eq. (2).

Integration of Eqs. (5) through (7) yields a solution which describes the secular orbit divergence of two initially adjacent points for \(\Delta \omega_f / \Delta \phi > 1\):

\[
\begin{align*}
\frac{\langle \phi_\phi^2 \rangle(t)}{\Delta \phi^2} &= \left[ \frac{\psi_\phi^2}{\Delta \psi_0^2} + \frac{\theta_\phi^2}{\Delta \theta^2} + \frac{\phi_\phi^2}{\Delta \phi^2} \right] \times \left\{ 1 + \frac{\Omega_\phi \exp[i \omega_f t_0]}{\Delta \omega_f + i \omega_f} \frac{1}{\Delta \omega_f + i \omega_f} \right\} + \frac{\theta_\phi^2}{\Delta \phi^2} \frac{1}{\Delta \theta^2} \left\{ \frac{\phi_\phi + \Omega_\phi \exp[i \omega_f t_0]}{\Delta \omega_f + i \omega_f} \right\} \left\{ \frac{1}{\Delta \omega_f + i \omega_f} \right\} \exp[i \omega_f t].
\end{align*}
\]

At this juncture, we define the eddy lifetime as the time necessary for two adjacent points to diverge to characteristic eddy size, i.e.,

\[
\langle \phi_\phi^2 \rangle(t)/\Delta \phi^2 = 1,
\]

\[
\tau_{\text{eddy}} \equiv \Delta \omega_f^{-1} \ln(\cdots)^{-1},
\]

where \(\cdots\) is the expression multiplying \(\exp\Delta \omega_f t\) on the right hand side (RHS) of Eq. (8). We recall that Eq. (8) implies \(\cdots < 1, \cdots = 1\) defines the physical extent of the eddy. It is usually in an ellipsoidal shape which is deformed due to the flow shear.

Here, the time dependence of the \(E \times B\) flow shear induces the wave-like structure to the eddy shape as indicated by the imaginary component in \(\cdots\). Practically, the most crucial modification is the reduction of the radial size due to \(E \times B\) flow shear which is usually accompanied by the reduction of the fluctuation level. Therefore, we define the radial correlation length with a phase parameter \(t_0\) for which the eddy shape is close to a simple deformed ellipsoid with a minimum wave-like structure. This happens when

\[
\text{Im} \left( \frac{\Omega_\phi \exp[i \omega_f t_0]}{\Delta \omega_f + i \omega_f} \right) = 0.
\]

The radial correlation length \(\Delta r_0 = \Delta \psi_0 / R B_\phi\), is reduced by the flow shear relative to its value \(\Delta r_0 = \Delta \psi_0 / R B_\phi\), determined by ambient turbulence alone:

\[
\left( \frac{\psi_\phi}{\Delta \psi_0} \right)^2 + 1 = 1 + \frac{\omega_{\text{eff}}^2}{\Delta \omega_f^2} = 1 + \frac{\omega_{\text{eff}}^2}{\Delta \omega_f^2}.
\]

Here, \(\omega_{\text{eff}} = \omega_E^{(0)} (1 + 3 F^2 + 4 F^2)^{1/4} / (1 + F) \sqrt{1 + 4 F} \). We have used Eq. (11) to eliminate \(t_0\) from Eq. (12). When \(E_r\) varies
slowly enough such that \( F < 1 \), we have \( \omega_{E\text{ff}} \approx \omega_E^{(0)} \), and recover the previous result in general toroidal geometry.\(^{18}\) On the other hand, when \( \omega_{E\text{ff}} \) varies fastly in time such that \( F > 1 \), we have \( \omega_{E\text{ff}} < \omega_E^{(0)} \). In this case, it is difficult to achieve turbulence suppression. The ratio \( \omega_{E\text{ff}} / \omega_E^{(0)} \) is plotted as a function of \( \omega_f / \Delta \omega_T \) in Fig. 1.

### III. SIMULATION RESULTS

In this section, we discuss the possible relevance of our analytical results to the recent gyrofluid\(^{13}\) and gyrokinetic\(^{27}\) simulations. Toroidal gyrofluid simulations showed that ITG turbulence can drive fluctuating sheared \( \mathbf{E} \times \mathbf{B} \) flows which play an important role in saturating the turbulence.\(^{19,21,26}\) These flows are typically of radial size \( k_r \rho_i \sim 0.1 \), but consist of a broad \( k_r \) spectrum. Since the \( \mathbf{E} \times \mathbf{B} \) shearing rate is proportional to \( k_r^2 \Phi \), the high \( k_r \) component of \( \Phi \), although small in magnitude, can contribute significantly to the \( \mathbf{E} \times \mathbf{B} \) shearing rate.\(^{33}\) The instantaneous \( \mathbf{E} \times \mathbf{B} \) shearing rate which varies in radius and time is much higher than the maximum linear growth rate for a significant portion of the simulation domain. An example is shown in Fig. 2. In this flux-tube simulation, representative parameters of DIII-D\(^{36}\) high confinement mode (H-mode) core plasmas have been used: \( R_0/L_T = 6.9, \ \eta_i = L_n/L_T = 3.2, \ q = 1.4, \ \tilde{s} = (r/q)(dq/dr) = 0.78, \ T_i/T_e = 1, \) and \( \epsilon = a/R_0 = 0.36, \) where \( R_0 \) is the major radius, \( L_T \) and \( L_n \) are the temperature and density gradient scale lengths, respectively, \( T_i \) is the ion temperature, and \( T_e \) is the electron temperature. The simplified physics model included a circular cross section, no impurities, and an adiabatic electron response excluding the flux surface averaged part of potential. This proper electron adiabatic response has been found to be essential in obtaining enhanced zonal flow amplitude in ITG turbulence.\(^{20,37}\) This zero electron response to the flux surface averaged potential yields no radial particle flux, and has been previously adopted in ITG mode simulation which exhibits system size radial scale fluctuation driven \( \mathbf{E} \times \mathbf{B} \) flow.\(^{38}\)

Using gyrofluid simulation zonal flow spectrum and time history to calculate correlation time of zonal flows, we have estimated the effective shearing rate in Eq. (13) for each \( k_r \).

It has a broad peak at low to intermediate \( k_r \) and becomes smaller at high \( k_r \) as shown in Fig. 3. Since their contributions add up incoherently, we compare them with the maximum linear growth rate divided by the square root of number modes. Overall, they are comparable. This seems qualitatively consistent with considerable reduction, not complete suppression of turbulence observed in simulations.

![Figure 1](image1.png)

**FIG. 1.** \( \omega_{E\text{ff}} / \omega_E^{(0)} = ((1 + 3F)^3 + 4F^3)^{1/3} / (1 + F)^{1/2} \) in Eq. (13) is plotted as a function of \( \omega_f / \Delta \omega_T \) in Eq. (13), demonstrating the reduction of effective shearing rate as the zonal flows oscillate faster than the turbulent decorrelation time.

![Figure 2](image2.png)

**FIG. 2.** Instantaneous potential (a), and instantaneous shearing rate (b), associated with small-scale turbulence generated flow from gyrofluid simulations are plotted. In (b), instantaneous shearing rate is dominated by high \( k_r \) components. Maximum linear growth rate for this run is \( \gamma = 0.1 \omega_f / L_n \).

![Figure 3](image3.png)

**FIG. 3.** \( \omega_{E\text{ff}} / \omega_E^{(0)} \) in Eq. (13) is evaluated from nonlinear gyrofluid simulation data for each \( k_r \), and compared to the maximum linear growth rate denoted in a straight line. The flow correlation frequency estimated from simulation is used for \( \omega_f \). The zonal flow potential and the instantaneous shearing rate are also plotted.
Gyrokinetic toroidal code\textsuperscript{27} solves the toroidal nonlinear gyrokinetic Vlasov–Maxwell system\textsuperscript{39,40} in magnetic coordinate.\textsuperscript{41} This single code can be used in both global and annulus geometry simulations. Inclusion of zonal flows in simulations significantly reduce the steady state ion thermal transport as reported earlier.\textsuperscript{27} One of the new significant findings from this simulation is a broadening of $k_r$ spectrum of turbulence due to self-consistently generated zonal flows as shown in Fig. 4. These are in qualitative agreements with Eq. (12) which shows the reduction of radial correlation length due to the time varying $\mathbf{E} \times \mathbf{B}$ flow shear.

The instantaneous $\mathbf{E} \times \mathbf{B}$ shearing rate which is dominated by high $k_r$ components varies roughly on the turbulence time scale as shown in Fig. 5. It is much larger than the maximum linear growth rate for a significant portion of the simulation domain. These global simulations used representative parameters of DIII-D H-mode core plasmas, which have a peak ion temperature gradient at $r=0.5a$ with the aforementioned local parameters used in the flux-tube gyrofluid simulations. The size of the plasma column was $a=160\rho_i$, where $\rho_i$ is the thermal ion gyroradius measured at $r=0.5a$. ITG modes are unstable with these parameters. A parabolic $q$ profile has been used.

Our gyrofluid and gyrokinetic simulations share many similar features in the physics of zonal flows, despite their differences in simulation methods, simulation domains, and boundary conditions. However, the following quantitative difference between them deserves attention. Short wavelength components of zonal flows are more prominent in flux-tube gyrofluid simulations compared to the gyrokinetic simulations. Since the high $k_r$ components contribute significantly to the instantaneous $\mathbf{E} \times \mathbf{B}$ shear rate, the peak value of the instantaneous shearing rate from the gyrofluid simulation shown in Fig. 2 is about twice the value obtained from the gyrokinetic simulation shown in Fig. 5. However, according to the estimation from nonlinear gyrofluid simulation shown in Fig. 3, most of turbulence shearing is done by the low to intermediate $k_r$ part of zonal flows. Since the long wavelength components of zonal flows are more prominent in gyrokinetic simulations compared to the flux–tube gyrofluid simulations, we can speculate that higher value of steady state ion thermal diffusivity typically observed in gyrofluid simulation in comparison to that seen in a gyrokinetic simulation could be partially due to an underestimation of the low $k_r$ component of zonal flows linearly undamped by collisionless neoclassical process\textsuperscript{42} by the original gyrofluid closure. Modifications to gyrofluid closures are currently being investigated to improve the quantitative comparisons\textsuperscript{33} and more details will be reported elsewhere.

IV. DISCUSSIONS

Although we have focused our discussions on the zonal flows in toroidal ITG-turbulence in this paper, zonal flows have been widely studied in fluid mechanics community as recently summarized.\textsuperscript{43} Furthermore, their possible existence has been theoretically predicted\textsuperscript{44} for the Hasegawa–Mima system\textsuperscript{45} which is isomorphic to the Rossby wave equation.\textsuperscript{46} Finally, it should be noted that turbulence driven flows have been observed in the nonlinear simulations of other turbulence models,\textsuperscript{24,47–51} although their radial scales were typically of the order of a fraction of the simulation domain.

On the experimental side, the increase of the $\mathbf{E} \times \mathbf{B}$ shearing rate\textsuperscript{19} from experimentally measured profiles\textsuperscript{1–3,5–14} has shown robust semi-quantitative agreement with the suppression of turbulence and confinement improvement. We note that both theories and experiments deal with flow time scales which are much slower than the turbulent time scales. One of the fastest time variations of macroscopic radial electric field observed in the core of magnetic confinement experiment to our knowledge is Heavy Ion Beam Probe (HIBP) measurements of dynamic electric field bifurcation in the Compact Helical System (CHS).\textsuperscript{52} Even in this experiment, a typical time scale of electric field pulsation is of the order of hundreds of microseconds which is much slower than the typical eddy turn-over time of turbulent plasmas. Therefore, the previous formulations in various geometry\textsuperscript{17,18,32,53,54} assuming the slow evolution of $E_r$ seem still appropriate in addressing the transition to the enhanced confinement regime associated with the macroscopic radial electric field.

The experimental evidence of density fluctuations with characteristics related to the zonal flows has been obtained...
from the DIII-D edge.\textsuperscript{55} Density fluctuations with extremely small $k_{|}$ and finite $k_{\parallel}$ have been observed via Phase Contrast Imaging (PCI) technique. The correlation function of these fluctuations is in good agreement with the recent gyrofluid results,\textsuperscript{33} although it remains to be seen whether zonal flows actually exist at the core. The development of new diagnostics with finer spatio-temporal resolution capable of measuring zonal flows would contribute significantly to our understanding of core turbulent transport.

Finally, we discuss the limitations of our simple analytical model. The zonal flows in the simulation contain spectrum of components with various frequencies and radial wavelengths. We have considered the effect of each component separately in our kinematic approach. While our theory treats only the ambient fluctuations statistically, it would be desirable to have a statistical description of the fluctuating flows as well. One recent approach to this is to consider the case where zonal flows vary more smoothly in space and time than drift wave turbulence such that an action invariant can be defined and the eikonal approximation is justified. Then, a wave-kinetic theory in the context of drift wave propagation in the presence of random media consisting of zonal flows can be pursued.\textsuperscript{56} The predicted broadening of the $k_{\parallel}$ spectrum of turbulence via random refraction is also in qualitative agreement with our gyrokinetic simulation results.\textsuperscript{27}

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