A conservative scheme of drift kinetic electrons for gyrokinetic simulation of kinetic-MHD processes in toroidal plasmas.

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A conservative scheme of drift kinetic electrons for gyrokinetic simulations of kinetic-MHD processes in toroidal plasmas has been formulated and verified. Both vector potential and electron perturbed distribution function are decomposed into adiabatic part with analytic solution and non-adiabatic part solved numerically. The adiabatic parallel electric field is solved directly from the electron adiabatic response, resulting in a high degree of accuracy. The consistency between electrostatic potential and parallel vector potential is enforced by using the electron continuity equation. Since particles are only used to calculate the non-adiabatic response, which is used to calculate the non-adiabatic vector potential through the Ohm’s law, the conservative scheme minimizes the electron particle noise and mitigates the cancellation problem. Linear dispersion relations of kinetic Alfvén wave and collisionless tearing mode in cylindrical geometry have been verified in gyrokinetic toroidal code (GTC) simulations, which show that perpendicular grid size can be larger than electron collisionless skin depth when the mode wavelength is longer than the electron skin depth.

I. Introduction

The excitation and evolution of macroscopic electromagnetic instabilities in magnetized plasmas often depend on kinetic effects at microscopic scales as well as the nonlinear coupling of multiple physical processes, which span disparate spatial and temporal scales. For example, the excitation of neoclassical tearing mode (NTM), the most likely instability leading to disruption in tokamak [1], depends on nonlinear interaction of magnetohydrodynamic (MHD) instability, microturbulence, collisional (neoclassical) transport, and energetic particle effects. NTM islands flatten the local pressure profile and modify plasma flow, thus affecting microturbulence and the neoclassical bootstrap current. On the other hand, microturbulence can impact island dynamics by regulating plasma current and electron heat conductivity along and across the magnetic field, and by driving sheared flows via Reynolds stress and Maxwell stress. Energetic particles also strongly
affect the tearing modes. A fully self-consistent NTM simulation must incorporate nonlinear
interactions between resistive MHD tearing modes, neoclassical transport, microturbulence,
energetic particle effects. These kinetic-MHD processes all have characteristic frequencies below
ion cyclotron frequency, which can be most efficiently studied by gyrokinetic simulation model
[2-4].

Gyrokinetic particle simulation [4-7] has emerged as a powerful tool for studying nonlinear
physics of low frequency kinetic-MHD processes, thanks to advances in physics models,
numerical algorithms, and computing power. In particular, the formulation of a perturbative (δf)
simulation method [8-9] has drastically reduced particle noises. Nonetheless, electromagnetic
gyrokinetic particle simulation incorporating simultaneously ion and electron dynamics is
numerically challenging due to the small electron-to-ion mass ratio, especially for the long
wavelength modes in high β (ratio of kinetic to magnetic pressure) plasmas [10, 11]. A
split-weight scheme [12, 13] has thus been developed to analytically calculate the electron
adiabatic responses to the parallel electric field, which further reduces electron noises.

However, the parallel electric field can be insignificant for long wavelength shear Alfvén
waves with non-tearing parity (such as kinetic ballooning modes and Alfvén eigenmodes),
where electrostatic parallel electric field nearly cancels out with the inductive parallel electric
field when the mode polarization is close to the ideal MHD in high-β (ratio of kinetic to
magnetic pressure) plasmas. A small error in calculating the electrostatic and vector potentials
could result in a large error in the parallel electric field, which could greatly affect the electron
dynamics. The problem is worse for the tearing mode, which is driven by a parallel electric
field in a narrow resonant layer with a width of an electron collisionless skin depth. The
electron responses to this parallel electric field is not adiabatic since k_{||} ≈ 0. Outside this
narrow tearing layer, the parallel electric field is small and the electron response is close to the
ideal MHD response in the form of massless (adiabatic) electrons, which carry a non-resonant
current that induces the Alfvén waves. A numerical difficulty is to recover this non-resonant
current from the electron distribution function when parallel electric field is very small in the
ideal MHD limit.

Another well-known numerical difficulty in electromagnetic gyrokinetic simulations is
that the calculation of the inductive parallel electric field requires a time-derivative of the
parallel vector potential, an operation often leading to a numerical instability. To avoid this
explicit time-derivative operation, a popular practice is to use canonical momentum as an
independent velocity variable (p_{||}-formulation), which artificially adds two large terms to the
original Ampere’s law. Analytically, these two terms should cancel exactly with each other.
However, a small error in numerically evaluating these two terms in the Ampere’s law can
give rise to a residue that leads to a large error in the parallel vector potential [14, 15], which
is known as “cancellation problem” in some gyrokinetic particle and continuum codes that
requires sophisticated numerical techniques or reduced physics models [15-28]. Furthermore, to mitigate the cancellation problem in the electromagnetic simulations, the perturbative ($\delta f$) method or split-weight scheme needs to use a small perpendicular grid size of the collisionless electron skin depth even for simulations of non-tearing modes such as electromagnetic ion temperature gradient (ITG) instability [26].

To overcome the difficulties of simultaneously treating the dynamics of ions and electrons in electromagnetic simulations, a reduced fluid-kinetic hybrid electron model [22-25] was developed by expanding the electron drift kinetic equation using the electron-to-ion mass ratio as a small parameter. In the lowest order, electron is adiabatic and becomes a massless fluid. The electron kinetic effects are incorporated in the higher order kinetic equation. The important technical here is to calculate the parallel electric field directly from electron parallel force balance, rather than from the cancellation between electrostatic and inductive fields. Another key technique is to calculate the non-resonant current from the adiabatic response by using the parallel vector potential, rather than from the electron distribution function that suffers from electron particle noises. This model accurately recovers low frequency plasma dielectric responses and faithfully preserves linear and nonlinear wave-particle resonance for non-tearing modes in the simulations of microturbulence and Alfvén eigenmodes using gyrokinetic toroidal code (GTC) [29-39]. Maximum numerical efficiency is achieved by overcoming the electron Courant condition and suppressing tearing modes (and associated electron noises). The fluid-kinetic hybrid electron model solves the original Ampere’s law, which is free from the cancellation problem. Recently, the hybrid model has been extended to incorporate the tearing modes by adding the resistivity to electron momentum equation for the resistive tearing mode [40] and by implementing the finite-mass electron fluid model [41] for the collisionless tearing mode [42].

Nonetheless, it is desirable to develop a unified formulation that solve the exact electron drift kinetic equation to incorporate low frequency electromagnetic fluctuations with both tearing parity and non-tearing parity on the same footing in the gyrokinetic simulations of nonlinear interactions of multiple kinetic-MHD processes in high-$\beta$ plasmas. This ambitious builds on our recent work [43], where we show that the perturbed electron density and current measured from kinetic markers in conventional gyrokinetic simulations do not satisfy the electron continuity equation in conventional $\delta f$ scheme due to the electron particle noises. Consequently, the electrostatic potential calculated from the density and the parallel vector potential calculated from the current are not consistent with each other, which results in an unphysically large parallel electric field. This inconsistency is the primary cause of the well-known numerical difficulty of electromagnetic $\delta f$ simulation of long wavelength MHD modes with kinetic electrons as shown by our theoretical error analysis. To overcome this inconsistency problem, only the highest order moment (either electron flow or pressure) required to close the electron fluid system is calculated by using the distribution function, and other lower order moments are calculated from conservative moment equations of the drift
kinetic equation. This new electron scheme, which is referred to as a “conservative scheme” [43], solves the exact drift kinetic equation. For example, our scheme in the $p_{||}$-formulation uses the continuity equation to time advance the electron density perturbation, and only the perturbed canonical flow is calculated from perturbed distribution function [43]. In the $v_{||}$-formulation (using parallel velocity as an independent velocity variable), we need to use both electron continuity and momentum equations, and the electron kinetic effects come into the system through the electron pressure.

In this work, the original fluid-kinetic hybrid electron model is extended to solving the exact drift kinetic electron model using the conservative scheme [43] in the $v_{||}$-formulation. In this new conservative scheme, the electron density perturbation is calculated from the continuity equation, the electron flow is calculated from the vector potential by inverting the Ampere’s law, and the vector potential is calculated from the Ohm’s law. Only the highest order moment, i.e., electron pressure is calculated from the guiding center distribution function, which is used to close the system. The electron perturbed distribution function $\delta f$ is decomposed into adiabatic part and non-adiabatic part $\delta f = \delta f_a + \delta h$. We further separate the parallel vector potential $A_{||}$ into adiabatic part and non-adiabatic part $A_{||} = \delta A_{||}^A + A_{||}^{NA}$. The adiabatic vector potential $\delta A_{||}^A$ and the adiabatic distribution $\delta f_a$ are defined and analytically solved self-consistently with each other. The non-adiabatic perturbed distribution function $\delta h$ can then be calculated in the simulation. By using the total Ohm’s law for $A_{||}$ integrated from drift kinetic equation and the analytic solution of $\delta A_{||}^A$, the Ohm’s law for solving $A_{||}^{NA}$ can be derived and solved in the simulation. In summary, our conservative scheme solves the adiabatic parallel electric field directly from the electron adiabatic response. The non-adiabatic parallel electric field is solved from Ohm’s law using electron non-adiabatic distribution function, which is free from the cancellation problem. The perturbed density and parallel flow are numerically conserved thanks to the use of continuity equation and Ohm’s law, which brings numerical stability for long time simulation.

The conservative scheme guarantees the conservation properties of electron perturbed density and parallel flow, thus, the consistency between the electrostatic potential and parallel vector potential. Since only the non-adiabatic electron response is calculated by using particles, our conservative scheme minimizes the electron particle noise like the split-weight scheme. Furthermore, the field equation for adiabatic vector potential is free from the cancellation problem, while the generalized Ohm’s law is only used to solve the non-adiabatic vector potential by using...
the non-adiabatic electron response, which mitigates the cancellation problem. We will show that the perpendicular grid size can be much larger than the electron skin depth for simulations of long wavelength modes. The linear dispersion relations of kinetic Alfvén wave (KAW) in uniform plasmas and collisionless tearing mode in cylindrical geometry have been verified in simulations using GTC. This conservative scheme can faithfully capture the tearing mode physics, and will be utilized for the gyrokinetic simulation of nonlinear interaction of multiple kinetic-MHD processes in toroidal plasmas ranging from micro-tearing mode \[45\] to neoclassical tearing mode.

The paper is organized as follows: The electromagnetic gyrokinetic model with \(v_{||}\)-formulation and the perturbative \( \delta f \) simulation method are described in Sec. II. In Sec. III, we formulate the conservative scheme of the drift kinetic electron model for nonlinear electromagnetic simulations in toroidal geometry. The verifications of this scheme for simulations of kinetic Alfvén wave and collisionless tearing mode are shown in Sec. IV. Conclusions are drawn in Sec. V.

II. Electromagnetic gyrokinetic simulation model

A. Nonlinear gyrokinetic equations and field equations

The gyrokinetic model has been widely used to study low frequency waves and instabilities in tokamak plasmas. The following gyrokinetic ordering is adopted in this paper:

\[
\frac{\omega}{\Omega_i} \sim \frac{\delta f}{f} \sim \frac{e \phi}{T} \sim \frac{\delta B}{B_0} \sim k_i \rho_i \sim O(\varepsilon_g) \ll 1,
\]

where \( \omega \) and \( \Omega_i = eB_0/cm_i \) are physical mode frequency and ion cyclotron frequency, \( m_i \) is ion mass, \( v_{th} = \sqrt{T/m_i} \) is the ion thermal velocity, \( \rho_i = v_{th}/\Omega_i \) is the ion gyro-radius, \( \delta f \) and \( f \) are perturbed and total particle distributions, \( \phi \) and \( \delta B \) are perturbed electrostatic potential and perturbed magnetic field, and \( k_{||} \) is parallel wave vector. This gyrokinetic ordering does not assume short perpendicular wavelength and thus can be used for simulations of long wavelength modes in the toroidal geometry. This ordering can be extended to allow equilibrium gradient scale length on the order of the perpendicular wavelength \[44\].

The following gyrokinetic Vlasov equation describes the gyrocenter dynamics by using gyrocenter position \( R \), magnetic moment \( \mu \) and parallel velocity \( v_{||} \) as independent variables.
in the five dimensional phase space [2]:

\[
\left( \frac{\partial}{\partial t} + \mathbf{R} \cdot \nabla + \dot{v}_\parallel \frac{\partial}{\partial v_\parallel} \right) f_\alpha (\mathbf{R}, v_\parallel, \mu, t) = 0 ,
\]

(1)

\[
\dot{R} = v_\parallel \mathbf{b}_0 + v_\parallel \frac{\delta \mathbf{B}}{B_\parallel} + \mathbf{v}_e + \mathbf{v}_d + \mathbf{v}_{\text{NL}},
\]

(2)

\[
\dot{v}_\parallel = -\frac{1}{m_\alpha B_\parallel} \left( Z_\alpha \nabla \phi + Z_\alpha \nabla \psi_{\text{NL}} + \mu \nabla B_0 \right) - \frac{Z_\alpha}{cm_\alpha} \frac{\partial A_\parallel}{\partial t},
\]

(3)

where \( Z_\alpha \), \( m_\alpha \) and \( f_\alpha \) represent the charge, mass and distribution of \( \alpha = i, e \) particle species, respectively. \( A_\parallel \) is perturbed parallel vector potential, \( \mathbf{B}_0^* = \mathbf{B}_0 + \left( \mathbf{B}_0 v_\parallel / \Omega_\alpha \right) \nabla \times \mathbf{b}_0 \), \( \mathbf{B}_\parallel = \mathbf{b}_0 \cdot \mathbf{B}_0^* \), \( \delta \mathbf{B} = \nabla \times \left( \mathbf{A}_0 - \mathbf{b}_0 \right) \) and \( \mathbf{B}^* = \mathbf{B}_0^* + \delta \mathbf{B} \). The nonlinear potential \( Z_\alpha \psi_{\text{NL}} = \frac{\mu}{2B_0^2} \left( \mathbf{B}^* \right)^2 - \frac{m_\alpha}{2} \left( \mathbf{v}_e + v_\parallel \mathbf{B}_0 \right)^2 \) is given in the drift kinetic limit [2]. \( \Omega_\alpha = \frac{Z_\alpha B_0}{cm_\alpha} \) is cyclotron frequency. The overbar \( \overline{\cdots} = \frac{1}{2\pi} \int d\mathbf{x} d\mathbf{\xi} (\cdots) \delta \left( \mathbf{R} + \mathbf{p}_\alpha - \mathbf{x} \right) \) represents gyro-average, \( \mathbf{\xi} \) is gyrophase angle, \( \mathbf{x} \) is particle position and \( \mathbf{p}_\alpha = \frac{\mathbf{b}_0 \times \mathbf{v}_e + \mathbf{v}_{\text{NL}}}{\Omega_\alpha} \) is gyroradius.

For the drift kinetic electron, the gyro-average can be removed, and \( \mathbf{R} = \mathbf{x} \) in Eqs. (1)-(3). The \( \mathbf{v}_e \) and \( \mathbf{v}_{\text{NL}} \) are perturbed \( \mathbf{E} \times \mathbf{B} \) drift due to the electrostatic potential and nonlinear potential, respectively, and \( \mathbf{v}_d \) is magnetic drift:

\[
\overline{\mathbf{v}_e} = \frac{c}{B_\parallel} \mathbf{b}_0 \times \nabla \phi ,
\]

\[
\mathbf{v}_{\text{NL}} = \frac{c}{B_\parallel} \mathbf{b}_0 \times \nabla \psi_{\text{NL}} ,
\]

and

\[
\mathbf{v}_d = \frac{cm_\alpha v_\parallel^2}{Z_\alpha B_\parallel^*} \mathbf{b}_0 \times \left( \mathbf{b}_0 \cdot \nabla \mathbf{b}_0 \right) + \frac{cm_\alpha}{Z_\alpha B_\parallel^*} \mathbf{b}_0 \times \nabla B_0 .
\]

The appearance of \( \mathbf{B}_\parallel^* \) ensures that the gyrocenter equations of motion preserve the Hamiltonian structure and satisfy the Liouville’s theorem [2]:
The Eq. (4) in the conservative form is equivalent to Eqs. (1)-(3).

The electrostatic potential \( \phi \) is solved by the gyrokinetic Poisson’s equation:

\[
\frac{Z_i^n_e}{T_i} (\phi - \overline{\phi}) = Z_i \overline{n}_i - en_e .
\]

where \( \overline{\phi}(\mathbf{x},t) = \frac{1}{n_i} \int dv f_{i0}(\mathbf{R},v_{||},\mu,t) \overline{\phi}(\mathbf{R},t) \) is the second gyrophase-averaged potential, \( f_{i0} \) is the ion equilibrium distribution, \( \overline{n}_i(\mathbf{x},t) = \int dv f_i(\mathbf{R},v_{||},\mu,t) \) and \( n_e(\mathbf{x},t) = \int dv f_e(\mathbf{x},v_{||},\mu,t) \) are the gyrophase-averaged ion and electron densities, \( \int dv = \frac{2\pi}{m_i} \int B_{||} d\nu_{||} d\mu \frac{1}{2\pi} \int \delta(\mathbf{R} + \mathbf{p}_a - \mathbf{x}) d\mathbf{R} d\xi \). The electrostatic potential \( \phi \), ion density \( \overline{n}_i \) and electron density \( n_e \) contain both zonal and non-zonal components. We can solve Eq. (5) for both zonal and non-zonal components together or we can solve them separately.

The parallel vector potential \( A_{||} \) is solved by the parallel Ampere’s law:

\[
\nabla_{||}^2 A_{||} = -\frac{4\pi}{c} (T_{||} + J_{||}) .
\]

where \( T_{||}(\mathbf{x},t) = Z_i \int v_{||} d\nu f_i(\mathbf{x},v_{||},\mu,t) \) and \( J_{||}(\mathbf{x},t) = q_e \int v_{||} d\nu f_e(\mathbf{x},v_{||},\mu,t) \).

Eqs. (1)-(3), (5) and (6) form a closed system with the \( v_{||} \)-formulation, sometime called “symplectic representation”, for electromagnetic gyrokinetic simulations of low frequency waves and instabilities in magnetized plasmas. The perpendicular Ampere’s law can be added to incorporate the compressional magnetic perturbations in the above model [46].

**B. Perturbative \( \delta f \) simulation scheme**

To reduce particle noises, a perturbative (\( \delta f \)) simulation scheme [8, 9] has been developed and successfully exercised in gyrokinetic simulations by splitting the total distribution function into the equilibrium and perturbed parts \( f_\alpha = f_{\alpha0} + \delta f_\alpha \). Only the perturbed distribution function \( \delta f_\alpha \) is dynamically calculated in the simulation, which reduces the numerical noise by a factor of
\((\delta f_\alpha / f_\alpha)^2\). The equilibrium distribution \(f_{\alpha 0}\) is defined as:

\[
L_0 f_{\alpha 0} = 0 ,
\]

(7)

where \( L_0 = \frac{\partial}{\partial t} + (\mathbf{v}_l B_0 + \mathbf{v}_\alpha) \cdot \nabla - \frac{\mu}{m_\alpha B_{\alpha \parallel}} B_0^\parallel \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \) is the equilibrium propagator. The equilibrium distribution function \(f_{\alpha 0}\) in the toroidal geometry is a neoclassical solution to Eq. (7) [47]. In the \(\delta f\) simulation method, the neoclassical solution \(f_{\alpha 0}\) to Eq. (7) is implicitly built-in, but can be solved in the gyrokinetic simulation using the \(\delta f\) method [48, 49]. The equilibrium distribution function \(f_{\alpha 0}\) then appears as a source for the perturbed \(\delta f_\alpha\) equation, where \(f_{\alpha 0}\) can then be approximated as a shifted Maxwellian as routinely used in analytic theory and turbulence simulation: \(f_{\alpha 0} = n_{\alpha 0} \left( \frac{m_\alpha}{2\pi T_{\alpha 0}} \right)^{3/2} \exp \left[ -\frac{m_\alpha (v_{\parallel 0} - u_{\parallel 0})^2 + 2\mu B}{2T_{\alpha 0}} \right] \), where \(u_{\parallel 0}\) is the parallel equilibrium flow of each species, and equilibrium Ampere’s law is satisfied:

\[
Z n_\alpha v_{\parallel 0} - e n_e v_{\parallel 0} = \frac{c}{4\pi} \mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0 , \quad n_{\alpha 0} \quad \text{is the equilibrium density and} \quad T_{\alpha 0} \quad \text{is the equilibrium temperature of each species. The effects of neoclassical transport on microturbulence can be incorporated by adding the neoclassical source term in the \(\delta f_\alpha\) equation.}
\]

Using Eq. (7) to subtract Eq. (1), the equation of \(\delta f_\alpha\) can be derived as:

\[
L \delta f_\alpha = - (\delta L_1 + \delta L_2) f_{\alpha 0} ,
\]

(8)

where \( L = \frac{\partial}{\partial t} + \mathbf{R} \cdot \nabla + \dot{\mathbf{v}}_{\parallel} \frac{\partial}{\partial v_{\parallel}} \) is the total propagator, \(\delta L_1\) and \(\delta L_2\) are the linear and nonlinear perturbed propagators:

\[
\delta L_1 = \left( v_{\parallel l} \nabla \mathbf{B}_{\parallel} + \mathbf{v}_\alpha \right) \cdot \nabla - \left[ \frac{\mu}{m_\alpha B_{\parallel}} \mathbf{B}_{\parallel} \cdot \nabla B_{\parallel} + \frac{Z_\alpha}{m_\alpha B_{\parallel}} \left( \mathbf{B}_{\parallel} \cdot \nabla \phi + \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} \right) \right] \frac{\partial}{\partial v_{\parallel}} ,
\]

\[
\delta L_2 = \mathbf{v}_{\parallel l} \cdot \nabla - \frac{Z_\alpha}{m_\alpha B_{\parallel}} \mathbf{B}_{\parallel} \cdot \nabla \phi \frac{\partial}{\partial v_{\parallel}} - \frac{Z_\alpha}{m_\alpha B_{\parallel}} \mathbf{B}_{\parallel} \cdot \nabla \psi_{\parallel} \frac{\partial}{\partial v_{\parallel}} .
\]

Here we only keep nonlinear terms up to the second order in \(\delta L_2\).
In the simulation, we can rewrite Eq. (8) by defining the particle weight as \( w_a = \delta f_a / f_a \):

\[
\frac{dw_a}{dt} = (1 - w_a) \left[ -\left( \vec{v}_k + \vec{v}_{NL} \right) \right] \frac{\nabla f_a}{f_a} + \left( \frac{\mu}{B} \vec{\delta B} + Z_a \frac{\vec{B}^*}{B} \cdot \nabla \phi + Z_a \frac{\vec{\partial A}}{c} + \frac{1}{m_a f_a} \right) \frac{\psi_{NL}}{\nabla \psi_{NL}} \frac{1}{m_a f_a} \frac{\partial f_a}{\partial v_{\parallel}} \tag{9}
\]

In principle, Eqs. (1)-(3), (5), (6) and (9) can be used as a closed system for the perturbative simulations. However, due to the small electron-ion mass ratio, it is difficult to apply these equations directly for the electromagnetic simulations, especially for the long wavelength MHD modes. The time derivative of the parallel vector potential \( \partial A_{\parallel} / \partial t \) in the \( v_{\parallel} \)-formulation is also difficult to evaluate by a finite difference method. To overcome these difficulties, we will formulate in the next section the conservative scheme of the drift kinetic electron with the \( v_{\parallel} \)-formulation for the gyrokinetic simulations of kinetic-MHD processes in toroidal geometry.

### III. Conservative scheme for drift kinetic electron

In this section, the subscript “e” in most of electron quantities is omitted for simplicity since our formulation of the conservative scheme is only for drift kinetic electrons. To simplify electron equations for our conservative scheme, we use the following auxiliary ordering, which can be relaxed if necessary,

\[
\frac{k_{\perp}}{k_{\parallel}} \sim \frac{1}{k_{\perp} L} \sim O(\varepsilon) < 1,
\]

where \( k_{\perp} \) is perpendicular wave vector, and \( L \) is plasma equilibrium scale length.

\[
L \sim \left( L_n = \nabla \ln n, L_T = \nabla \ln T, L_B = \nabla \ln B_0 \right) (T \text{ is plasma temperature}, n \text{ is electron density, and } B_0 \text{ is equilibrium magnetic field} ).
\]

The small parameter \( \varepsilon = a/R \) is the inverse aspect ratio, where \( a \) is the minor radius and \( R \) is the major radius. We keep all linear terms of the electron drift kinetic equation (i.e., both tearing parity and non-tearing parity) and nonlinear terms up to the order \( O(\varepsilon \varepsilon) \) in Eq. (8). The terms related to the nonlinear potential \( \psi_{NL} \) in Eq. (8) are the order \( O(\varepsilon^2 \varepsilon) \) and thus dropped for both ion and electron species since \( |\psi_{NL}|/|\phi| \sim O(\varepsilon) \). This ordering is valid for long wavelength MHD modes with a low toroidal mode number.

The electrostatic potential \( \phi \) is separated into the non-zonal part \( \delta \phi \) and the zonal part.
(flux-surface averaged part) $\langle \phi \rangle$ as:

$$\phi = \delta \phi + \langle \phi \rangle,$$

where $\langle \cdots \rangle$ represents the flux surface averaging, $\theta$ and $\zeta$ are the poloidal and toroidal angles in magnetic coordinates, and $J$ is the Jacobian. The zonal and non-zonal components can be solved by Eq. (5) together, or solved separately by using the non-zonal and zonal solvers in practice due to the large difference of the amplitude between non-zonal and zonal components in nonlinear simulation [1, 24, 36]. The parallel vector potential $A_\parallel$ consists of the adiabatic part $\delta A^a_\parallel$ and non-adiabatic part $A^{na}_\parallel$ as:

$$A_\parallel = \delta A^a_\parallel + A^{na}_\parallel.$$

The adiabatic part $\delta A^a_\parallel$ only contains the non-zonal component associated with electron adiabatic responses and the non-resonant current as in the fluid-kinetic hybrid electron model [22]. The non-adiabatic part $A^{na}_\parallel$ contains both the zonal component $\langle A^{nak}_\parallel \rangle$ and non-zonal component $\delta A^{nak}_\parallel$ due to electron non-adiabatic responses and nonlinear ponderomotive forces.

The electron distribution function consists of the equilibrium and perturbed parts as $f = f_0 + \delta f$, which are described by Eqs. (7) and (8), respectively. The equilibrium distribution provides the equilibrium density $n_0 = \int f_0 dv$, which does not evolve in the simulation. The perturbed distribution produces perturbed density $\delta n = \int \delta f dv$, which evolves in the simulation. We further separate the perturbed distribution $\delta f$ into the adiabatic response $\delta f_a$ and non-adiabatic response $\delta h$ as $\delta f = \delta f_a + \delta h$.

The adiabatic vector potential $\delta A^a_\parallel$ is defined as:

$$\frac{\partial \delta A^a_\parallel}{\partial t} = c_0 \cdot \nabla \delta \phi_{nd}.$$

(10)

where the non-zonal field $\delta \phi_{nd}$ is defined as:
\[
\frac{e\delta \phi_{md}}{T_0} = \frac{\delta n - \langle \delta n \rangle}{n_0} - \frac{e\delta \phi}{T_0} - \frac{\partial n_0}{\partial \psi} \frac{\delta \psi^A}{n_0} - \frac{\partial n_0}{\partial \alpha} \frac{\delta \alpha^A}{n_0},
\]

where perturbed electron density is separated into a zonal part \( \langle \delta n \rangle \) and non-zonal part \( \delta n - \langle \delta n \rangle \). \( \psi \) is the poloidal flux, and \( \alpha = q(\psi)\theta - \zeta \) is the magnetic field line label. \( \delta \psi^A \) and \( \delta \alpha^A \) are the perturbed quantities. Note that we use electron non-zonal density perturbation to define the adiabatic vector potential through Eqs. (10) and (11).

The adiabatic magnetic perturbation \( \delta B^A \) is defined as:

\[
\delta B^A = \nabla \times (\delta A^A_0 b_0),
\]

Meanwhile, we can write the equilibrium magnetic field and adiabatic magnetic perturbation using the Clebsch representation:

\[
B_0 = \nabla \psi_0 \times \nabla \alpha_0,
\]

and

\[
\delta B^A = \nabla \psi_0 \times \nabla \delta \alpha^A + \nabla \delta \psi^A \times \nabla \alpha_0,
\]

From Eqs. (10), (12)-(14), we obtain the equations for \( \delta \psi^A \) and \( \delta \alpha^A \):

\[
\frac{\partial \delta \psi^A}{\partial t} = -c \frac{\partial \delta \phi_{md}}{\partial \alpha_0},
\]

\[
\frac{\partial \delta \alpha^A}{\partial t} = c \frac{\partial \delta \phi_{md}}{\partial \psi_0},
\]

The adiabatic distribution \( \delta f_a \) is defined by the following equation:

\[
v_{||} b_0 \cdot \nabla \delta f_a = -v_{||} \frac{\delta B^A}{B_0} \cdot \nabla f_0 \Bigg|_0 - \left[ \frac{\mu}{B_0} \delta B^A \cdot \nabla B_0 + q a_b_0 \cdot \nabla \left( \delta \phi + \delta \phi_{md} \right) \right] v_{||} f_0.
\]

Using Eqs. (13) and (14), we can solve Eq. (17) and derive \( \delta f_a \) as:

\[
\delta f_a = \frac{e(\delta \phi + \delta \phi_{md})}{T_0} f_0 + \frac{\partial f_0}{\partial \psi_0} \bigg|_{v_{||}} \frac{\delta \psi^A}{\partial \alpha_0} + \frac{\partial f_0}{\partial \alpha_0} \bigg|_{v_{||}} \delta \alpha^A.
\]

Integrating Eq. (18) in the velocity space and define adiabatic density response \( \delta n_a = \int \delta f_a dv \), we have:
We note that Eq. (19) indicates that \( \langle \delta n_a \rangle = 0 \). From Eqs. (11) and (19), we have:

\[
\delta n = \delta n_a + \langle \delta n \rangle .
\]  \hspace{1cm} (20)

Eq. (20) indicates that the adiabatic response \( \delta f_a \) contributes to all non-zonal density perturbation, and the non-adiabatic response \( \delta h \) contributes to all zonal density perturbation, i.e.,

\[
\int \delta h d\mathbf{v} = \langle \delta n \rangle .
\]

Note that although \( \delta h \) does not contribute to non-zonal density perturbation, it contributes to higher order non-zonal moments such as perturbed flow and pressure. We note that Eqs. (10) and (18) ensure that the adiabatic vector potential \( \delta A_{||}^A \) and the adiabatic distribution \( \delta f_a \) are defined and analytically solved self-consistently with each other, an important consistency absence in the split-weight scheme where adiabatic response is defined by using the total vector potential.

In order to calculate the total perturbed density \( \delta n \), we take the moment of Eq. (4) to obtain the electron continuity equation as:

\[
\frac{\partial \delta n}{\partial t} + \nabla \cdot \left[ n_0 \left( \delta u_{|| e} \mathbf{b}_0 + \mathbf{V}_e + u_{|| e} \right) \frac{\delta \mathbf{B}}{B_0} \right] + \frac{1}{T_0} \left( \delta \mathbf{P}_\perp + \delta \mathbf{P}_|| \right) + \frac{n_0 \delta u_{|| e}}{B_0} \delta \mathbf{B} + \delta n \mathbf{V}_e = 0 ,
\]  \hspace{1cm} (21)

where the equilibrium part is removed by using Eq. (6). \( \delta u_{|| e} \) and \( u_{|| e} \) are the perturbed and equilibrium parallel velocities of electron guiding center, and \( \delta \mathbf{P}_|| \) and \( \delta \mathbf{P}_\perp \) are the perturbed parallel and perpendicular pressures of electron guiding center. \( \mathbf{V}_e = c \mathbf{b}_0 \times \nabla \phi / B_0 \) is \( \mathbf{E} \times \mathbf{B} \) drift, \( \mathbf{V}_e = c \mathbf{b}_0 \times \left( \mathbf{b}_0 \cdot \nabla \mathbf{b}_0 \right) \), and \( \mathbf{V}_e = c \mathbf{b}_0 \times \nabla B_0 \). \( \delta \mathbf{B} = \delta \mathbf{B}^A + \delta \mathbf{B}^{NA} \) is the total magnetic perturbation, and \( \delta \mathbf{B}^{NA} = \nabla \times \left( A_{|| e}^A \mathbf{b}_0 \right) \) is the non-adiabatic part of the magnetic perturbation. The continuity equation (Eq. (21)) together with Ampere’s law and quasi-neutrality condition can recover the MHD vorticity equation in uniform plasmas\(^\cite{52}\).

The values of \( \delta u_{|| e} \), \( \delta \mathbf{P}_|| \) and \( \delta \mathbf{P}_\perp \) are required to evolve Eqs. (21). The \( \delta u_{|| e} \) term can be calculated by inverting the Ampere’s law Eq. (6) as:
\[ \delta u_{\parallel e} = \frac{c}{4\pi e n_0} \nabla_{\perp}^2 A_{\parallel} + \frac{Z_i}{e} \delta \tilde{u}_{\parallel} . \]  

(22)

Here, we use the original Ampere’s law to derive the parallel electron flow \( \delta u_{\parallel e} \). The adiabatic part of the parallel vector potential \( \delta A_{\parallel}^A \) is solved analytically by Eqs. (10) and (11), and the non-adiabatic part of the parallel vector potential \( A_{\parallel}^{\text{NA}} \) is solved from the non-adiabatic electron response \( \delta h \), which will be described later. This method is free from the cancellation problem and greatly improves the accuracy of calculating the parallel electric field, especially for the long wavelength modes.

The electron parallel and perpendicular pressures consists of adiabatic and non-adiabatic parts as:

\[ \delta P_{\parallel} = \int_{\Omega} m_e v_{\parallel}^2 (\delta f_e + \delta h) d\mathbf{v} = \delta P_{\parallel}^A + \delta P_{\parallel}^{\text{NA}} , \]

\[ \delta P_{\perp} = \int_{\Omega} \mu B_0 (\delta f_e + \delta h) d\mathbf{v} = \delta P_{\perp}^A + \delta P_{\perp}^{\text{NA}} . \]

We take the moment of Eq. (18) to get the adiabatic electron pressures as:

\[ \delta P_{\parallel}^A = e n_0 \left( \delta \phi + \delta \phi_{\text{ad}} \right) + \frac{\partial (n_e T_0)}{\partial \psi_0} \delta \psi^A + \frac{\partial (n_e T_0)}{\partial \alpha_q} \delta \alpha^A , \]  

(23)

\[ \delta P_{\perp}^A = e n_0 \left( \delta \phi + \delta \phi_{\text{ad}} \right) + \frac{\partial (n_e T_0)}{\partial \psi_0} \delta \psi^A + \frac{\partial (n_e T_0)}{\partial \alpha_q} \delta \alpha^A . \]  

(24)

The non-adiabatic electron pressures are calculated from the kinetic electron response as:

\[ \delta P_{\parallel}^{\text{NA}} = \int m_e v_{\parallel}^2 \delta h d\mathbf{v} , \]  

(25)

\[ \delta P_{\perp}^{\text{NA}} = \int \mu B_0 \delta h d\mathbf{v} . \]  

(26)

When removing the non-adiabatic parts of the parallel vector potential and electron responses, Eqs. (10)-(12), (15)-(16) and (21)-(24) form a massless fluid electron model, which is identical to the lowest order adiabatic electron in the fluid-kinetic hybrid electron model [22-25].

Next, we formulate the equations for non-adiabatic kinetic electron response \( \delta h \) and non-adiabatic parallel vector potential \( A_{\parallel}^{\text{NA}} \). Using the relation \( \delta f = \delta f_e + \delta h \) and Eq. (8), the equation for the non-adiabatic electron response \( \delta h \) can be written as:

\[ L \delta h = -\delta L_f f_0 - L_0 \delta f_e - \delta L_2 f_0 - \left( \delta L_1 + \delta L_2 \right) \delta f_e , \]  

(27)

where term \{I\} is linear and term \{II\} is nonlinear.

Defining the weight function as: \( w_e = \delta h / f_e \), one can write the electron weight equation for the electron non-adiabatic response:
$$
\frac{dw_e}{dt} = \frac{1-w_e}{1+\varepsilon} \left[ -\delta L_1 f_0 - L_0 \delta f_0 - \delta L_2 f_0 - (\delta L_1 + \delta L_2) \delta f_0 \right] \tag{28}
$$

By using the relations of Eqs. (18)-(20), the equation for the time derivative of adiabatic response \( \partial \delta f_a / \partial t \) can be obtained as:

$$
\frac{1}{f_0} \frac{\partial \delta f_a}{\partial t} = \frac{1}{n_0} \left[ \frac{\partial \delta n}{\partial t} - \delta \left( \frac{\partial n}{\partial t} \right) \right] + \frac{1}{f_0} \frac{\partial f_a}{\partial T_0} \frac{\partial T_0}{\partial n} + \frac{1}{f_0} \frac{\partial \delta v^a}{\partial t} + \frac{1}{f_0} \frac{\partial \delta \psi^a}{\partial t} + \frac{1}{f_0} \frac{\partial \delta a^a}{\partial t}, \tag{29}
$$

which is required in evolving Eq. (28). The time derivative of the adiabatic response \( \partial \delta f_a / \partial t \) can be calculated from Eqs. (15), (16) and (21) easily. In contrast, the time derivative needs to be solved from a higher order moment equation in the split-weight scheme [13, 26].

In order to close the system, we can solve the non-adiabatic vector potential \( A_{||}^{\text{NA}} \) using Ohm’s law. To obtain the Ohm’s law, we need the electron momentum equation, which can be integrated from the electron drift kinetic equation (Eq. (4)) in the conservative form:

$$
\begin{align*}
&n_0 \frac{\partial \delta u_{||e}}{\partial t} + \nabla \cdot \left[ n_0 \delta u_{||e} \left( V_e + 3V_e + V_e \right) + n_0 u_{||P} V_e \right] \\
&+ \frac{q}{m_e} n_0 \left[ \left( B_0 + \frac{\delta B}{B_0} \right) \cdot \nabla \phi + \frac{1}{c} \frac{\partial A_{||}}{\partial t} \right] + \frac{1}{m_e} \nabla \cdot \left[ \delta P_{||} \left( \frac{B_0}{B_0} + \frac{\delta B}{B_0} \right) + P_{||} \frac{\delta B}{B_0} \right], \tag{30}
\end{align*}
$$

where we only keep the terms up to the second order \( O(\varepsilon^2) \) and make a closure and truncation on the parallel energy fluxes: \( \int m_e v_{||} \delta f dv = n_0 T_0 \delta u_{||e} \) and \( \int \nu_{||B_0} \delta f dv = 3n_0 T_0 \delta u_{||e} \). The truncated term is \( \int \nabla \cdot \left[ \nu_{||B_0} \delta f_{||} \right] v_{||} dv \), which is much smaller than the leading order term \( \int \nabla \cdot \left[ v_{||B_0} \delta f \right] v_{||} dv \). Here, we note that Eq. (30) with closure and truncation is only used below to derive the Ohm’s law for the non-adiabatic vector potential. Eq. (30) is not used to calculate the electron parallel flow \( \delta u_{||e} \), which is calculated by using the exact Ampere’s law (Eq. (22)).

Substituting the Ampere’s law Eq. (22) into electron momentum Eq. (30), we can derive the Ohm’s law equation for total vector potential \( A_{||} \). Subtracting the Ohm’s law for \( A_{||} \) by Eq. (10) and using Eqs. (23)-(26), we can derive the equation for the non-adiabatic vector potential \( A_{||}^{\text{NA}} \):
where

\[
\begin{align*}
\chi_{||} &= -\frac{c}{en_0} \mathbf{b}_0 \cdot \nabla \delta P_{||}^{\text{NA}} - \frac{c}{en_0 B_0} \mathbf{B}^{\text{NA}} \cdot \nabla \delta P_{||}^{\text{NA}} - \frac{c}{en_0 B_0} \mathbf{B}^{\text{NA}} \cdot \nabla \delta \phi_{\text{ind}} \\
&\quad + \frac{c}{B_0} \mathbf{B}^{\text{NA}} \cdot \nabla \phi - \frac{cm_e}{en_0} \nabla \left[ n_0 \delta u_{\parallel,\|} \left( 3 \mathbf{V}_e + \mathbf{V}_e \right) + n_0 u_{\|,\|} \mathbf{V}_e \right] - \frac{cm_e}{en_0} \nabla \cdot \left( n_0 \delta u_{\parallel,\|} \mathbf{V}_e \right), \\
&\quad + \frac{c}{en_0} \frac{P_{\|}}{B_0^2} \mathbf{B}^{\text{NA}} \cdot \nabla B_0 - \frac{c}{en_0} \frac{\delta \mathbf{P}_{\|}^{\text{NA}} - \delta \mathbf{P}_{\perp}^{\text{NA}}}{B_0^2} \mathbf{B}_0 \cdot \nabla B_0
\end{align*}
\]

We drop the ion contribution \( \partial \delta \mathbf{U}_{||}/\partial t \) in Eq. (31) due to the small electron-to-ion mass ratio.

In Eq. (31), terms \{I\} and \{II\} represent the electron inertia. Terms \{III\}, \{IV\} and \{V\} are the pressure gradient terms associated with either kinetic electron response or non-adiabatic perturbed magnetic field. Term \{VI\} is the leading drive of the zonal current generated by drift Alfvén waves [50]. Term \{VII\} is the zonal flow effect on non-zonal current. Terms \{VIII\} and \{IX\} are linear and nonlinear convective motions, respectively. We compare the ordering of \{IX\} with \{VI\} as \( O\left( \frac{IX}{VI} \right) \sim k^2 d_e^2 \), thus term \{IX\} is as important as \{VI\} for the physics containing \( k^2 d_e^2 \sim 1 \) scale such as collisionless tearing mode. Term \{X\} represents the mirror force contribution. When we set \( \delta P_{||}^{\text{NA}} = \delta P_{\perp}^{\text{NA}} = 0 \), i.e., removing the kinetic electron effects, Eq. (31) is equivalent to Eq. (7) from Liu and Chen [41], and this drift kinetic electron model reduces to the finite-mass fluid electron model [41]. Eq. (31) solves the non-adiabatic vector potential \( A_{||}^{\text{NA}} \) from the non-adiabatic perturbed distribution \( \delta h \). The perpendicular Laplacian on the left-hand-side can be ignored for long wavelength non-tearing modes. For collisionless tearing mode, the wavelength is on the order of the electron skin depth anyway. Therefore, Eq. (31) is free from the cancellation problem.

Eqs. (1)-(3), (5), (9), (10)-(12), (15)-(16), (18), (21)-(26), (28)-(29) and (31) form a closed system for electromagnetic simulation with kinetic electrons. It can be seen that all the electron kinetic effects come into our system from the perturbed kinetic pressures, and the lower order moments are calculated from the moment equations, thus it guarantees the conservation properties of perturbed density and parallel flow through Eqs. (10), (21), (22) and (31) as well as the consistency between the electrostatic potential and vector potential (ion parallel flow contribution to the vector potential is much smaller than electron parallel flow). It will be shown that the perpendicular grid size does not need to resolve the electron skin depth when the wavelength is longer than the electron skin depth in the simulation of KAW in section IV. The difference
between the conservative scheme and fluid-kinetic hybrid electron model [22-25] is that the fluid-kinetic hybrid electron model solve the electron drift kinetic equation approximately, while the conservative scheme solve the exact electron drift kinetic equation.

IV. Verification of conservative scheme

To verify the conservative scheme for electromagnetic simulations with kinetic electrons, we have implemented it in the gyrokinetic toroidal code (GTC) [7], and carried out simulations of the KAW in uniform plasmas and the collisionless tearing mode in the cylindrical geometry with magnetic shear. In this section, the quantities with subscript “e” represent the electron quantities.

A. Kinetic Alfvén wave in uniform plasmas

Assuming that ion only provides the polarization density in uniform plasmas, we first use the conservative scheme to derive the linear dispersion relation. Eq. (27) can be simplified in linear and uniform plasmas as:
\[
\left( \frac{\partial}{\partial t} + v_i b_0 \cdot \nabla \right) \delta h = - \frac{e}{m_e} \left( b_0 \cdot \nabla \delta \phi + \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} \right) \frac{\partial f_{\delta e}}{\partial v_{\perp}} - \left( \frac{\partial}{\partial t} + v_i b_0 \cdot \nabla \right) \delta f_0. \tag{32}
\]
In the long wavelength limit, the gyrokinetic Poisson equation (5) reduces to
\[
\frac{c^2}{4\pi e V_{\perp}^2} \nabla_\perp^2 \phi = \delta n_e. \tag{33}
\]
In uniform plasmas, Eqs. (11), (18), (21), (22) and (31) reduce to
\[
en_e T_e \left( \delta \phi + \delta \phi_{\parallel e} \right) = \delta n_e T_e, \tag{34}
\]
\[
\delta f_0 = \frac{e \left( \delta \phi + \delta \phi_{\parallel e} \right)}{T_e} f_{\delta e}, \tag{35}
\]
\[
\frac{\partial}{\partial t} \frac{\delta n_e}{n_0} + n_0 b_0 \cdot \nabla \delta u_{\parallel e} = 0, \tag{36}
\]
\[
\delta u_{\parallel e} = \frac{c}{4\pi e n_e} \nabla_\perp^2 A_{\parallel}, \tag{37}
\]
\[
\left( \nabla_\perp^2 - \frac{\omega_{pe}^2}{c^2} \right) \frac{\partial A_{\parallel e}}{\partial t} = -c n_0 B_0 \cdot \nabla \delta \phi_{\parallel e} - \frac{\omega_{pe}^2}{en_e c} b_0 \cdot \nabla \delta P_{\parallel e}. \tag{38}
\]
Applying the Fourier transform to Eqs. (10) and (32)-(38) and considering \( A_\parallel = \delta A_{\parallel e} + A_{\parallel e} \) :
\[
\delta, \to -i \omega, \quad b_0 \cdot \nabla \to ik_{\parallel}, \quad \text{and} \quad \nabla_\perp \to ik_\perp, \quad \text{the linear dispersion relation of KAW based on this model in the uniform plasmas is:}
\]
\[
\left( \frac{\omega^2}{k^2 V^2_a} - 1 \right) \left[ 1 + \xi_e Z(\xi_e) \right] = k^2 \rho^2_s,
\]

(39)

where \( \xi_e = \omega / \sqrt{2 k T_e / m_e} \), \( \rho_s = C_s / \Omega_{ci} \), \( V_{the} = \sqrt{T_e / m_e} \), \( C_s = \sqrt{T_e / m_i} \), and \( Z(\xi_e) \) is the plasma dispersion function:

\[
Z(\xi_e) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{e^{-t^2}}{t - \xi_e} dt.
\]

In the simulation of KAW in a uniform plasma, the electron temperature is \( T_{e0} = 5.0keV \) and magnetic field is \( B_0 = 1.5T \), the ratio between parallel and perpendicular wave vectors is \( k_{||}/k_\perp = 0.01 \). We scan the electron density and thus the value of \( \beta_e = 8 \pi n_{e0} T_{e0} / B_0^2 \). Firstly we use \( n_{e0} = 1.0 \times 10^{13} \text{ cm}^{-3} \) and the corresponding \( \beta_e = 0.9\% \), and verify the dependence of the frequency on the wavevector \( k \rho_s \) as well as the perpendicular grid size \( \Delta x / d_e \). The simulation results agree well with the analytic theory as shown by Fig. 1, which indicates that the perpendicular grid size can be much bigger than the electron skin depth. Secondly, we fix the wavevector \( k \rho_s = 0.48 \), and change the electron density from \( n_{e0} = 1.0 \times 10^{13} \text{ cm}^{-3} \) to \( n_{e0} = 2.0 \times 10^{14} \text{ cm}^{-3} \), and verify the dispersion relations of KAW for different \( \beta_e \) values. As shown by Fig. 2, both the frequency and damping rate agree with analytic theory very well when \( \beta_e \gg m_e / m_i \). In high \( \beta_e \) regime, the damping of KAW is too weak to be measured accurately, since the ion Landau damping is removed in the simulation.

These agreements between GTC simulations of the analytic theory verify that there is no constrain of the electron skin depth on the perpendicular grid size in our simulations. The simulations achieve high accuracy in both long wavelength and high \( \beta_e \) regimes, which demonstrates the numerical advantages of this model.
FIG. 1. Dependence of KAW frequency on wavelength (bottom) and perpendicular grid size (top) from GTC simulations and from analytic theory.

FIG. 2. Dependence of KAW frequency (a) and damping rate (b) on $\beta_e$ from GTC simulations and from analytic theory.

B. Collisionless tearing mode in cylindrical geometry

In this subsection, we verify the conservative scheme for theory and simulation of collisionless tearing mode in cylindrical geometry. For simplicity, we only keep the equilibrium current that drives the tearing mode, and neglect the density and temperature gradients which could contribute to the real frequency of the tearing mode. Therefore we would have purely growing collisionless tearing mode. From Eqs. (10), (25), (32), (34) and (38), one can readily derive the electron response to the tearing mode:
\[ \nabla_z^2 \omega A_i = \frac{\omega_p^2}{c^2} \left( \omega A_i - k_{||} \delta \phi \right) \tilde{Z} '(\tilde{\xi}_e), \tag{40} \]

where \( \tilde{Z} '(\tilde{\xi}_e) \) is the derivative of the plasma function with respect to \( \tilde{\xi}_e \).

For the collisionless tearing mode in the plasmas with uniform density and temperature profiles but non-uniform magnetic field, the continuity equation Eq. (21) reduces to

\[ \frac{\partial \delta n_e}{\partial t} + n_{e0} b_0 \cdot \nabla \delta u_{ie} + \frac{\delta B}{B_0} \cdot \nabla u_{ie} = 0, \tag{41} \]

where \( u_{ie} = -\frac{c}{4\pi\epsilon_0 n_{e0}} b_0 \cdot \nabla \times B_0 \) in the ion frame.

Combining Eqs. (33), (37), (40) and (41), one will have the eigenmode equation for the tearing mode:

\[ \nabla_z^2 \omega A_i = \nabla_z^2 \left[ \frac{1}{d_z^2} \omega_p^2 \tilde{Z} '(\tilde{\xi}_e) A_i - \frac{m_i}{m_e} k_{||} \tilde{\xi}_e \tilde{Z} '(\tilde{\xi}_e) (k_{||} \nabla_z^2 - k_{||}^* A_i, \tag{42} \right. \]

where \( k_{||} = \frac{d^2 k_{||}}{dr^2} \).

Following the asymptotic matching method of Drake and Lee [51], also Liu and Chen [41], one can derive the dispersion relation for the collisionless tearing mode. By noticing that in the inner region, near the mode rational surface \( k_{||} \sim 0 \), the mode structure is steeper (i.e. \( k_r \gg k_\theta \sim L_\theta \)) than that of outer ideal MHD region \( (k_r \sim k_\theta \sim L_\theta) \), so one only needs to keep the leading terms \{I\} and \{II\} in Eq. (42) for inner region equation:

\[ \nabla_z^2 A_{i||} = \frac{1}{d_z^2} \omega_p^2 \tilde{Z} '(\tilde{\xi}_e) A_{i||}, \tag{43} \]

where the subscript “i” means the inner region.

For the outer ideal MHD region away from the mode rational surface, term \{III\} will dominate, since the tearing mode frequency is smaller than the Alfvén wave frequency, so one...
can neglect the term \( \{\Pi\} \) for the outer region. Since the magnetic diffusive time due to the electron inertial term \( \{I\} \) is much longer than the tearing mode and Alfvén wave periods, the electron inertia related term \( \{I\} \) can also be removed. Thus, the outer region equation for \( A_{o\parallel} \) becomes:

\[
(k_{\|} \nabla_{\perp}^{2} - k_{\|}^\perp) A_{o\parallel} = 0,
\]

where the subscript “o” means the outer region.

Using the constant \( A_{\parallel} \) approximation, and matching the inner and outer region by using the boundary condition:

\[
\frac{A_{i\parallel}}{A_{o\parallel}} = \frac{A_{o\parallel}'}{A_{i\parallel}'} = \Delta',
\]

where \( A_{i\parallel}' = dA_{i\parallel}/dr \), \( A_{o\parallel}' = dA_{o\parallel}/dr \) and \( \Delta' = d\Delta/dr \).

One will have the dispersion relation for collisionless tearing mode from our kinetic model as:

\[
-i\omega = \gamma = \frac{d_{e}^{2}}{\sqrt{\pi}} |k_{\parallel}\nu_{te}| \Delta_o'.
\]

The kinetic dispersion relation, Eq. (46), derived from our conservative scheme is the same as the result from the electron drift kinetic equation by Drake and Lee [51].

When we set \( \partial P_{i\parallel}^{NA} = 0 \) in Eq. (38), i.e., remove electron kinetic effects but keep finite electron inertia, we can get the growth rate of the collisionless tearing mode from the finite-mass electron fluid model:

\[
-i\omega = \gamma = \frac{d_{e}^{2}}{\pi} |k_{\parallel}\nu_{te}| \Delta_o'.
\]

This fluid dispersion relation, Eq. (47), is the same with the result of Liu and Chen’s fluid model [41]. The origin for the difference of \( \sqrt{\pi} \) between Eq (46) and (47) comes from the fact that the finite mass electron fluid model assumes that the background electrons are two counter propagating cold beams with the same speed [41]. In the kinetic model, we use the Maxwellian distribution for background electron [51].
After verifying the analytic dispersion relation of our formulation, we now verify the GTC simulation of collisionless tearing mode using the conservative scheme. In order to compare with analytic theory, we simulate the collisionless tearing mode in the cylindrical geometry with magnetic shear. The equilibrium parameters are: uniform equilibrium electron density \( n_e = 1.0 \times 10^{12} \text{cm}^{-3} \) and temperature \( T_e = 5.0 \text{keV} \), axial magnetic field \( B_0 = 1.0T \), and the axial length is \( L = 2\pi R_0 \) where \( R_0 = 1.0m \). It should be pointed out that the radial grid size in this collisionless tearing mode simulation needs to resolve the electron skin depth \( d_e \) near the rational surface, since the mode structure of the collisionless tearing mode has a scale length of \( d_e \). This is different from the KAW simulation in the last subsection, where the radial grid size does not need to resolve the \( d_e \) scale. In the cylindrical geometry, we carry out the simulations of the collisionless tearing mode in both fluid and kinetic regimes by using our model with the \( q \) profile as shown by Fig. 3, which gives rise to an unstable collisionless tearing mode with \( m=2 \) and \( n=1 \). Firstly, we drop the second term on the RHS of Eq. (38), and our model reduces to the finite mass electron fluid model \([41, 42]\). The fluid simulation with the realistic electron-ion mass ratio \( m_e/m_i = 1/1837 \) gives the growth rate \( \gamma = 0.0014\left(C_s/R_0\right) \). The structures of this \((2, 1)\) mode for the parallel vector potential \( A_\parallel \) and electrostatic potential \( \phi \) on the poloidal plane are shown in Fig.4 (a) and (b). The fluid simulation results are verified by a 1D eigenvalue code \([41]\) solving the same fluid model, which gives the growth rate \( \gamma = 0.0015\left(C_s/R_0\right) \), and the mode structure agrees with the GTC fluid simulations. Secondly, we apply the exact Eq. (38) with the non-adiabatic pressure term in the kinetic electron simulation.

By using the same equilibrium parameter, the kinetic simulation gives the growth rate of the collisionless tearing mode \( \gamma = 0.0031\left(C_s/R_0\right) \). The mode structures of the parallel vector potential \( A_\parallel \) and electrostatic potential \( \phi \) on the poloidal plane are shown in Fig.4.
(c) and (d). The analytic growth rate from the electron drift kinetic equation is estimated as
\[ \gamma = 0.0027 \left( C_s / R_0 \right) \]
by multiplying \( \sqrt{\pi} \) to the fluid eigenvalue result according to Eqs. (46) and (47). GTC kinetic simulation results agree reasonably with the analytic theory. The small difference between the kinetic simulated and theoretical growth rate comes from the asymptotic method used in the analytic theory. The inner region is assumed to be infinitely narrow in the analytic theory, but has a finite width in the simulation. Finally, Fig. 5 shows that radial mode structures of \( A_\parallel \) between fluid eigenvalue code and GTC fluid simulation agree well, and that kinetic electrons have little effects on the radial mode structure. Thus, our model can faithfully capture the physics of collisionless tearing mode in both fluid and kinetic regimes.

![FIG. 3](image). Safety factor q profile in collisionless tearing mode simulation.
FIG. 4. Upper panels show the mode structures of (a) $A_{||}$ and (b) $\phi$ from GTC fluid electron simulations of (2,1) collisionless tearing mode in cylindrical geometry. Lower panels show mode structures of (c) $A_{||}$ and (d) $\phi$ from corresponding GTC kinetic electron simulations.

FIG. 5. Radial mode structures of $A_{\parallel}$ in (2, 1) collisionless tearing mode between GTC kinetic simulation, GTC fluid simulation, and fluid eigenvalue code in the cylindrical geometry.
V. Conclusions

In this paper, we present an efficient gyrokinetic ion and drift kinetic electron model for simulations of kinetic-MHD processes in the toroidal geometry. A novel conservative scheme is formulated for solving electrons drift kinetic equation. Both vector potential and electron perturbed response are decomposed into adiabatic and non-adiabatic parts. The adiabatic parts of vector potential and electron response are calculated analytically, while the non-adiabatic electron response is calculated by using the distribution function, and the non-adiabatic vector potential is solved using the non-adiabatic electron response through the Ohm’s law, which helps to decrease the particle noise and mitigate the cancellation problem in the simulation. Only the kinetic pressures are calculated from electron distribution function, while the lower order moments are calculated by using the moment equations, which guarantees the conservation properties of the electron perturbed density and parallel flow through the electron continuity equation and generalized Ohm’s law. Thus the consistency between the electrostatic potential and vector potential is enforced, which results in an accurate parallel electric field calculation.

Both continuity equation and Ohm’s law are derived by integrating the drift kinetic equation analytically, and the guiding center dynamics is described by the drift kinetic equation, thus this new conservative scheme solves the exact drift kinetic equation for electrons. The dispersion relation of kinetic Alfvén waves in high $\beta$ plasmas is verified in GTC simulation, which shows that the perpendicular grid size does not need to resolve the electron skin depth for the numerical accuracy and stability when the wavelength is longer than the electron skin depth. Finally, both the growth rate and mode structure of the collisionless tearing mode in GTC simulations are verified by analytic theory and eigenvalue calculation. The applications of the conservative scheme for gyrokinetic simulations of kinetic-MHD processes ranging from micro tearing mode to neoclassical tearing mode in toroidal plasmas will be reported in the future.

We note that a mixed-variable algorithm proposed by Mishchenko et al. [21] splits the vector potential into ideal MHD part and higher order perturbed part, where the ideal MHD part is free from the cancellation problem. In contrast, our conservative scheme is free from the cancellation problem since the adiabatic vector potential (including ideal MHD and adiabatic response associated with parallel electric field and pressure gradients) is calculated from the adiabatic parallel electric field. Our scheme also enforces the consistency between the electrostatic potential and parallel vector potential by using the electron continuity equation [43].

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$\Delta x / d_e$

$\omega / \Omega_{ci}$

$\beta_e = 0.9\%$

$1.0 / k \rho_s$