Particle diffusion in random fields: Time-nonlocal description and numerical simulations

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The theory of time-nonlocal random processes formulated in terms of the non-Markovian Fokker–Planck equation is used to describe the results of numerical simulations of particle diffusion in the random longitudinal field with given statistical properties. The simulations of particle motion were performed for the wide range of particle velocity and random field parameters. It is confirmed that conventional quasilinear theory in the approximation disregarding the time and velocity dependence of the diffusion coefficient in the velocity space can be used only in the case of small intensity and large width of turbulent field spectrum. The increase of the intensity as well as the decrease of the spectral width lead to considerable deviation of the results of simulations (such as saturation and frequent oscillation of the mean-square velocity displacement) from the predictions of the quasilinear theory. It is shown that in the case of small intensities these deviations can be successfully described in terms of non-Markovian generalization of the quasilinear approximation. In the case of high field intensity the description of these features would require more consistent account for the diffusion coefficient velocity dependence and time-nonlocal effects. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1525017]

I. INTRODUCTION

Recently much attention has been paid to the investigation of turbulent diffusion processes which manifest considerable deviation from classical diffusion. In many cases such deviations are associated with the non-Markoviety of the process under consideration.[1–10] It is typical of the time-nonlocal (non-Markovian) diffusion that the asymptotic value of the mean-square particle displacement (in both real and velocity spaces) is proportional to the time interval in the power $\alpha \neq 1$ which differs from the Einstein law for conventional diffusion ($\alpha = 1$). The non-Markovian effects can be introduced into the kinetic theory of transport processes in terms of various approaches. These are the continuous-time-random-walk theory (CTRW),[2,8,11] the decorrelation trajectory method,[12,13] the fractional Fokker–Planck equation method,[14,15] generalized transition probability approaches,[6] etc. More details about the current state of the problem are given in Refs. 8 and 10.

The existing theories, as well as numerical calculations within various models,[9,12] show that non-Markovian effects are definitely well-pronounced in the diffusion across external magnetic field generated by the drift-wave turbulence. In the case of one-dimensional Langmuir wave turbulence, the time-nonlocality of the turbulent transport is usually ignored. It is assumed that in this case the diffusion can be successfully described within the quasilinear theory or its various modifications. Recently, this theory has been applied to the description of particle diffusion in the prescribed set of longitudinal waves with random phases.[16] However, the comparison of the analytical predictions of the quasilinear theory and the numerical simulations shows that the agreement between the results can be achieved only under certain conditions. These conditions include restrictions of the evolution time interval, intensity, and the width of the turbulent field spectrum. For example, it was pointed out in Refs. 17 and 18 that time dependence of the mean-square velocity displacement for resonant particles is linear only for times less than some definite time instant that depends on the field intensity. For greater times, simulations show that the mean-square velocity displacement is saturated. It may be suggested that the discrepancy between the quasilinear theory and simulations even in the case of weak turbulence could be associated, on the one hand, with the effective particle friction due to the scattering by turbulent fields, and on the other hand, with considerable changes of particle distributions during the correlation time with respect to which the diffusion coefficient is introduced (the time-nonlocal effect).

The time-nonlocal generalization of the quasilinear theory and the Dupree–Weinstock renormalized theory was proposed in Refs. 5 and 6. The purpose of the present paper is to compare the predictions of the time-nonlocal kinetic theory in the case of Langmuir wave turbulence with the results of simulations and to specify the non-Markovian effects in the test particle diffusion in turbulent fields. We study particle motion in a random longitudinal field with known spectra which is not influenced by particle motion. The analytical description is performed on the basis of the theory proposed in Ref. 6. Basic equations used in this ap-

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proach are presented in Sec. II. Section III contains specific calculations of the particle transition probability and mean-square displacements. The description of the model for numerical simulations of particle diffusion and comparison of the results of simulations with the prediction of the theory are given in Sec. IV. Section V concludes the paper with a brief summary.

II. TIME-NONLOCAL KINETIC EQUATIONS

The kinetic equation for turbulent plasmas in the one-dimensional case can be obtained from the microscopic equations. It is not difficult to show that with the accuracy up to the time-nonlocality of the collision term, it can be written as[6]

$$\hat{L}^0 f(X,t)=\int_{-\infty}^{t} dt' \frac{\partial}{\partial v}\left\{\beta(t,t';v)v f(X,t') + \frac{\partial}{\partial v}(D(t,t';v)f(X,t'))\right\}, \tag{1}$$

where

$$\hat{L}^0=\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}+\frac{e}{m}\langle E\rangle\frac{\partial}{\partial v}, \tag{2}$$

$X\equiv(r,v)$, $\langle E\rangle$ is the regular electric field, if present, $\beta(t,t';v)$ and $D(t,t';v)$ are the kinetic coefficients expressed in terms of the turbulent field spectrum and particle transition probability,[6]

$$D(t,t';v)=\left(\frac{e}{m}\right)^2\int\frac{dk}{2\pi}\langle\delta E(t)\delta E(t')\rangle_k\langle e^{-ik\Delta r(X,t;t')}\rangle, \tag{3}$$

$$\beta(t,t';v)=\beta_1(t,t';v)+\beta_2(t,t';v), \tag{4}$$

$$\beta_1(t,t';v)=i\frac{4\pi e^2 n}{mv}\delta(t-t') \times\int\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\int dv'\frac{W^*_{k\omega}(v',v)}{k\varepsilon(k,\omega)}, \tag{5}$$

$$\varepsilon(k,\omega)=1-i\frac{4\pi e^2 n}{mk^2}\int dv\int dv' \times W_{k\omega}(v,v')k\frac{\partial f(X',t)}{\partial v'},$$

$$\beta_2(t,t';v)=-\frac{1}{v}\frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2\int\frac{dk}{2\pi}\langle\delta E(t)\delta E(t')\rangle_k \times\langle e^{-ik\Delta r(X,t;t')}\rangle, \tag{6}$$

$$W_{k\omega}(v,v')=\int_0^{\infty}d\tau e^{i\omega\tau}\int d\Delta r e^{-ik\Delta r} \times W(r+\Delta r,v;r,v';t+\tau,t), \tag{7}$$

$\Delta r(X',t';t)$ is the spatial displacement of the particle at time t which occupied the phase position X' at the initial time t', the superscript asterisk implies the complex conjugate.

The quantity $\langle e^{-ik\Delta r(X',t';t)}\rangle$ also can be expressed in terms of the particle transition probability in the phase space $W(X,X';t,t')\equiv W(r,v;r',v';t,t')$. We have

$$\langle e^{-ik\Delta r(X',t';t)}\rangle=\int d\Delta v\int d\Delta r W(r'+\Delta r,v'+\Delta v;r',v';t,t')e^{-ik\Delta r}. \tag{8}$$

Analogously,

$$\langle e^{-ik\Delta r(X,t;t')}\rangle=\int d\Delta v\int d\Delta r W(r+\Delta r,v+\Delta v;r,v,t',t)e^{ik\Delta r}. \tag{9}$$

The equation for the transition probability is similar to Eq. (1),[6]

$$\hat{L}^{(0)}W(X,X';t,t') =\int_{t'}^{t}dt''\frac{\partial}{\partial v}\left\{\beta(t,t'';v)vW(X,X';t'',t') + \frac{\partial}{\partial v}(D(t,t'';v)W(X,X';t'',t'))\right\}. \tag{10}$$

This equation should be supplemented by the initial condition

$$W(X,X';t',t')=\delta(X-X'). \tag{11}$$

In the particular case of small correlation times, $\tau_{\mathrm{cor}}\ll\tau_{\mathrm{rel}}$ (τ_{rel} is the characteristic time of the distribution changes), the quantities $f(X,t')$ and $W(X,X';t'',t')$ in the right-hand sides of Eqs. (1) and (10) can be replaced by $f(X,t)$ and $W(X,X';t,t')$, respectively, and we thus obtain the well-known equations with the collision terms in the Fokker–Planck form, i.e.,

$$\hat{L}^{(0)}f(X,t)=\frac{\partial}{\partial v}\left\{\beta(v)vf(X,t)+\frac{\partial}{\partial v}(D(v)f(X,t))\right\}, \tag{12}$$

$$\hat{L}^{(0)}W(X,X';t,t')=\frac{\partial}{\partial v}\left\{\beta(v)vW(X,X';t,t') + \frac{\partial}{\partial v}(D(v)W(X,X';t,t'))\right\}, \tag{13}$$

where

$$\beta(v)\equiv\beta=\int_{t'}^{t}dt''\beta(t,t'';v)\simeq\int_0^{\infty}dt''\beta(t,t'';v), \tag{14}$$

$$D(v)\equiv D=\int_{t'}^{t}dt''D(t,t'';v)\simeq\int_0^{\infty}dt''D(t,t'';v). \tag{15}$$

The transition to the Markovian limit could be also performed by a formal substitution

$$\beta(t,t';v)=\beta(v)\delta(t-t'),$$

$$D(t,t';v)=D(v)\delta(t-t'),$$

and thus in the Markovian approximation,

$$\beta(t,t';v)=\delta(t-t')\int_0^\infty dt''\beta(t,t'';v),$$

$$D(t,t';v)=\delta(t-t')\int_0^\infty dt''D(t,t'';v).$$

III. THE TRANSITION PROBABILITY AND MEAN-SQUARE DISPLACEMENTS

If both time and velocity dependencies of the kinetic coefficients can be ignored, we can find analytic solutions of the equations for the transition probabilities. In such cases the solution of Eq. (13) for the Markovian transition probability is given by

$$W_M(X,X';\tau)=\frac{e^{\beta\tau}}{2\pi\Delta^{1/2}}\times\exp\left\{-\frac{1}{2\Delta}(a\,\delta\rho^2+2h\,\delta\rho\,\delta P+b\,\delta P^2)\right\}, \tag{16}$$

where

$$\delta\rho=ve^{\beta\tau}-v',$$

$$\delta P=r-r'+\frac{v-v'}{\beta},$$

$$\Delta=ab-h^2, \tag{17}$$

$$a=\frac{2D\tau}{\beta^2},\quad b=\frac{D}{\beta}(e^{2\beta\tau}-1),\quad h=-\frac{2D}{\beta^2}(e^{\beta\tau}-1),$$

$$\tau=t-t'.$$

The solution for the non-Markovian case is given by[6]

$$W(X,X';\tau)=\int\frac{d\omega}{2\pi}e^{-i\omega\tau}\left[\int_0^\infty d\tau_1 e^{i\omega\tau_1}W_M(X,X';\tau_1)\right]_{\substack{D\to D_\omega,\\ \beta\to\beta_\omega}}, \tag{18}$$

where

$$D_\omega=\int_0^\infty d\tau e^{i\omega\tau}D(\tau;v),$$

$$\beta_\omega=\int_0^\infty d\tau e^{i\omega\tau}\beta(\tau;v). \tag{19}$$

Here we assume that in the case under consideration $\beta(t,t';v)\equiv\beta(\tau,v)$ and $D(t,t';v)\equiv D(\tau,v)$.

Substituting the transition probabilities (16) or (18) into Eqs. (3)–(9) we obtain a set of equations for the renormalized kinetic coefficients in the Markovian and time-nonlocal approximations, respectively. In particular, solution (18) generates the following time-nonlocal diffusion coefficient:

$$D(\tau,v)=\left(\frac{e}{m}\right)^2\int\frac{d\omega}{2\pi}e^{-i\omega\tau}\int_0^\infty d\tau_1 e^{i\omega\tau_1}\times\int\frac{dk}{2\pi}\langle\delta E(t)\delta E(t-\tau)\rangle_k\times\exp\left[\frac{ikv}{\beta_\omega}(1-e^{-\beta_\omega\tau_1})-\frac{k^2D_\omega}{\beta_\omega^2}\times\left(\tau_1-\frac{2}{\beta_\omega}(1-e^{-\beta_\omega\tau_1})+\frac{1}{2\beta_\omega}(1-e^{-2\beta_\omega\tau_1})\right)\right]. \tag{20}$$

The equation for the Markovian diffusion coefficient $D\equiv D(v)$ is given by integration $D(\tau,v)$ over the time variable τ,

$$D=\left(\frac{e}{m}\right)^2\int_0^\infty d\tau\int\frac{dk}{2\pi}\langle\delta E(t)\delta E(t-\tau)\rangle_k\times\exp\left[\frac{ikv}{\beta}(1-e^{-\beta\tau})-\frac{k^2D}{\beta^2}\left(\tau-\frac{2}{\beta}(1-e^{-\beta\tau})+\frac{1}{2\beta}(1-e^{-2\beta\tau})\right)\right]. \tag{21}$$

In the limit of small β_ω or β, Eqs. (20) and (21) reduce to

$$D(\tau,v)=\left(\frac{e}{m}\right)^2\int\frac{dk}{2\pi}\langle\delta E(t)\delta E(t-\tau)\rangle_k\times\int\frac{d\omega}{2\pi}e^{-i\omega\tau}\int_0^\infty d\tau_1\times\exp\left[i(\omega+kv)\tau_1-\frac{k^2D_\omega}{3}\tau_1^3\right] \tag{22}$$

and

$$D=\left(\frac{e}{m}\right)^2\int_0^\infty d\tau\int\frac{dk}{2\pi}\langle\delta E(t)\delta E(t-\tau)\rangle_k\times\exp\left[ikv\tau-\frac{k^2D}{3}\tau^3\right]. \tag{23}$$

Equation (23) gives the Dupree result for the renormalized diffusion coefficient. Having put $D=0$ in the right-hand-part of Eq. (23), we reproduce the well-known quasilinear result. Obviously, its time-nonlocal generalization is generated by Eq. (22),

$$D(\tau,v)=\left(\frac{e}{m}\right)^2\int\frac{dk}{2\pi}e^{ikv\tau}\langle\delta E(t)\delta E(t-\tau)\rangle_k. \tag{24}$$

The next step is to calculate the mean-square spatial and velocity displacements,

$$\langle\Delta v^2\rangle_\tau\equiv\langle\Delta v^2\rangle_{v\tau} = \int d\Delta r\int d\Delta v(\Delta v)^2 W(r+\Delta r,v+\Delta v;r,v;\tau) = \int\frac{d\omega}{2\pi}e^{-i\omega\tau}\int_0^\infty d\tau_1 e^{i\omega\tau_1}\left\{\frac{D_\omega}{\beta_\omega}(1-e^{-2\beta_\omega\tau_1}) + v^2 e^{-2\beta_\omega\tau_1}\right\}, \quad (25)$$

$$\langle\Delta r^2\rangle_\tau\equiv\langle\Delta r^2\rangle_{v\tau} = \int d\Delta r\int d\Delta v(\Delta r)^2 W(r+\Delta r,v+\Delta v;r,v;\tau) = \int\frac{d\omega}{2\pi}e^{-i\omega\tau}\int_0^\infty d\tau_1 e^{i\omega\tau_1} \times\left\{\frac{2D_\omega}{\beta_\omega^2}\left[\tau_1-\frac{2}{\beta_\omega}(1-e^{-\beta_\omega\tau_1}) + \frac{1}{2\beta_\omega}(1-e^{-2\beta_\omega\tau_1})\right]+\frac{v^2}{\beta_\omega^2}(1-e^{-\beta_\omega\tau_1})^2\right\}. \quad (26)$$

In the Markovian approximation, Eqs. (25) and (26) reduce to

$$\langle\Delta v^2\rangle_\tau=\frac{D}{\beta}(1-e^{-2\beta\tau})+v^2e^{-2\beta\tau}, \quad (27)$$

$$\langle\Delta r^2\rangle_\tau=\frac{2D}{\beta^2}\left[\tau-\frac{2}{\beta_\omega}(1-e^{-\beta\tau})+\frac{1}{2\beta}(1-e^{-2\beta\tau})\right] + \frac{v^2}{\beta^2}(1-e^{-\beta\tau}). \quad (28)$$

In the frictionless limit, $\beta_\omega\to 0$, we have

$$\langle\Delta v^2\rangle_\tau=2\int_0^\tau d\tau_1\tau_1 D(\tau-\tau_1;v), \quad (29)$$

$$\langle\Delta r^2\rangle_\tau=v^2\tau^2+\frac{1}{3}\int_0^\tau d\tau_1\tau_1^3 D(\tau-\tau_1;v). \quad (30)$$

At $\beta\to 0$, Eqs. (27) and (28) give

$$\langle\Delta v^2\rangle_\tau=2D\tau, \quad (31)$$

$$\langle\Delta r^2\rangle_\tau=\tfrac{1}{3}D\tau^3+v^2\tau^2. \quad (32)$$

In order to find the mean-square displacements explicitly, we have to know the fluctuation field spectrum. In what follows we use the model for the fluctuation potential,

$$\langle\delta\Phi(r,t)\delta\Phi(r',t')\rangle = \delta\Phi_0^2\exp\left[-\frac{\Delta k^2(r-r')^2}{4}\right] \times\cos(k_0(r-r')-\omega_0(t-t')), \quad (33)$$

i.e., we assume that the wave harmonics have the same frequency ω_0 and Gaussian amplitude distribution that makes a reasonable approximation for the Langmuir wave turbulence.

Equation (33) yields the electric field fluctuation spectrum given by

$$\langle\delta E(t)\delta E(t-\tau)\rangle_k = \delta\Phi_0^2 k^2\frac{\sqrt{\pi}}{\Delta k}\sum_{s=\pm 1}\exp\left[-\frac{(k-sk_0)^2}{\Delta k^2}\right]e^{-is\omega_0\tau}. \quad (34)$$

Thus, the time-nonlocal diffusion coefficient is of the form

$$D(\tau,v)=\left(\frac{e}{m}\right)^2\delta\Phi_0^2\frac{\sqrt{\pi}}{\Delta k}\sum_{s=\pm 1}e^{-is\omega_0\tau}\int\frac{d\omega}{2\pi}e^{-i\omega\tau} \times\int\frac{dk}{2\pi}k^2\exp\left[-\frac{(k-sk_0)^2}{\Delta k^2}\right] \times\int_0^\infty d\tau_1\exp\left[i\omega\tau_1+\frac{ikv}{\beta_\omega}(1-e^{-\beta_\omega\tau_1}) -\frac{k^2D_\omega}{\beta_\omega^2}\left(\tau_1-\frac{2}{\beta_\omega}(1-e^{-\beta_\omega\tau_1}) +\frac{1}{2\beta_\omega}(1-e^{-2\beta_\omega\tau_1})\right)\right]. \quad (35)$$

For the Fourier-component of the diffusion coefficient, one obtains

$$D_\omega=\left(\frac{e}{m}\right)^2\delta\Phi_0^2\frac{\sqrt{\pi}}{\Delta k}\sum_{s=\pm 1}\int\frac{dk}{2\pi}e^{-[(k-sk_0)^2/\Delta k^2]}k^2 \times\int_0^\infty d\tau e^{i(\omega-s\omega_0)\tau} \times\exp\left[ikvt_1(\tau)-\frac{k^2D_{\omega-s\omega_0}}{\beta_{\omega-s\omega_0}^2}t_2(\tau)\right], \quad (36)$$

where

$$t_1(\tau)=\frac{1}{\beta_{\omega-s\omega_0}}(1-e^{-\beta_{\omega-s\omega_0}\tau}), \quad (37)$$

$$t_2(\tau)=\tau-\frac{2}{\beta_{\omega-s\omega_0}}(1-e^{-\beta_{\omega-s\omega_0}\tau}) +\frac{1}{2\beta_{\omega-s\omega_0}}(1-e^{-2\beta_{\omega-s\omega_0}\tau}). \quad (38)$$

In the frictionless limit $\beta_\omega\to 0$, we have

$$t_1(\tau)=\tau, \quad (39)$$

$$t_2(\tau)=\frac{\beta_{\omega-s\omega_0}^2\tau^3}{3}. \quad (40)$$

After integration over k, Eq. (36) yields

$$D_\omega \simeq -\frac{1}{2v^2}\left(\frac{e}{m}\right)^2 \delta\Phi_0^2 \sum_{s=\pm 1} \int_0^\infty d\tau \exp\left[i(\omega - s\omega_0)\tau - \frac{k_0^2 D_{\omega - s\omega_0}}{\beta_{\omega - s\omega_0}^2} t_2(\tau)\right]\frac{\partial^2}{\partial t_1^2} \times \exp\left[isk_0 v t_1(\tau) - \frac{\Delta k^2 v^2 t_1^2(\tau)}{4}\right]$$
$$\simeq \frac{1}{2}\left(\frac{e}{m}\right)^2 \delta\Phi_0^2 k_0^2 \sum_s \int_0^\infty d\tau \times \exp\left[i(\omega - s\omega_0)\tau - \frac{\Delta k^2 v^2 t_1^2(\tau)}{4}\right] \times \exp\left[-\frac{k_0^2 D_{\omega - s\omega_0}}{\beta_{\omega - s\omega_0}} t_2(\tau) + isk_0 v t_1(\tau)\right]. \quad (41)$$

In the case of small ω (large evolution time), it is reasonable to put $\beta_{\omega - s\omega_0} \sim \beta_{\omega_0} = 0$ and thus

$$D_\omega \simeq -\frac{1}{2v^2}\left(\frac{e}{m}\right)^2 \delta\Phi_0^2 \sum_{s=\pm 1} \int_0^\infty d\tau e^{i(\omega - s\omega_0)\tau} \times \exp\left[-\frac{k_0^2 D_{\omega - s\omega_0}\tau^3}{3}\right]\frac{\partial^2}{\partial \tau^2} \times \exp\left[-\frac{\Delta k^2 v^2 \tau^2}{4} + isk_0 v\tau\right]. \quad (42)$$

Substituting Eq. (42) into Eq. (25) yields the mean-square velocity displacement, i.e.,

$$\langle \Delta v^2\rangle_\tau = \left(\frac{e}{m}\right)^2 \frac{\delta\Phi_0^2}{v^2} \sum_{s=\pm 1} \int_0^\infty d\tau_1 \int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega(\omega + 2i\beta_\omega)} \times \exp\left[i(\omega - s\omega_0)\tau_1 - \frac{k_0^2 D_{\omega - s\omega_0}\tau_1^3}{3}\right]\frac{\partial^2}{\partial \tau_1^2} \times \exp\left[-\frac{\Delta k^2 v^2 \tau_1^2}{4} + isk_0 v\tau_1\right]. \quad (43)$$

Here and in what follow we keep only the term generated by D_ω which describes mean-square velocity deviation from its mean value. For large τ and finite β_ω as $\omega \to 0$, Eq. (43) could be approximated by

$$\langle \Delta v^2\rangle_\tau = -\left(\frac{e}{m}\right)^2 \frac{\delta\Phi_0^2}{v^2} \sum_{s=\pm 1} \int_0^\tau d\tau_1 \frac{1}{2\beta_0} \times (1 - e^{-2\beta_0(\tau - \tau_1)}) \exp\left[-\frac{k_0^2 D_{-s\omega_0}\tau_1^3}{3}\right]\frac{\partial^2}{\partial \tau_1^2} \times \exp\left[-\frac{\Delta k^2 v^2 \tau_1^2}{4} + isk_0 v\tau_1\right]. \quad (44)$$

Equation (44) contains two quantities which are still unknown. These are the kinetic coefficients D_ω and β_ω. In the general case they should be calculated on the basis of Eq. (36) for D_ω and a similar equation for β_ω. The solutions of these equations would lead us to the time-nonlocal renormalized theory. In the present paper, however, we restrict ourselves to the case of weak turbulent fields, for which the kinetic coefficients can be estimated in terms of the perturbation theory. This means that the zeroth order approximation for D_ω is given by Eq. (36) with $D_\omega = 0$ in the right-hand part that reduces to

$$D_\omega \simeq \frac{1}{2}\left(\frac{e}{m}\right)^2 \delta\Phi_0^2 k_0^2 \sum_{s=\pm 1} \frac{i}{\omega - s\Omega + i0} \times \left[1 - W\left(\frac{\omega - s\Omega}{\Delta k v}\sqrt{2}\right)\frac{(\omega - s\omega_0)^2}{k_0^2 v^2}\right], \quad (45)$$

where $\Omega = \omega_0 - k_0 v$, $W(z)$ is the plasma dispersion function

$$W(z) = 1 - ze^{-(z^2/2)}\int_0^z dy e^{y^2/2} + i\left(\frac{\pi}{2}\right)^{1/2} ze^{-(z^2/2)}.$$

Concerning the quantity β_ω, we assume that in the case of turbulent plasmas, the polarization part of the friction coefficient is negligibly small $(\beta_1(t,t';v) = 0)$ and thus β_ω can be calculated as

$$\beta_\omega = -\frac{1}{v}\frac{\partial D_\omega}{\partial v}. \quad (46)$$

The latter formula generates the estimate in the zeroth order approximation, i.e.,

$$\beta_\omega = \frac{1}{2v^2}\left(\frac{e}{m}\right)^2 k_0^2 \delta\Phi_0^2 \sum_{s=\pm 1} \frac{i}{\omega - s\Omega + i0} \times \left\{ \left[1 - s\frac{\omega - s\omega_0}{k_0 v} + \frac{(\omega - s\omega_0)^2}{k_0^2 v^2}\right] - W\left(\frac{\omega - s\Omega}{\Delta k v}\sqrt{2}\right) \times \frac{(\omega - s\omega_0)^2}{\Delta k^2 v^2}\left(3 - \frac{(\omega - s\omega_0)(\omega - s\Omega)}{\Delta k^2 v^2}\right)\right\}. \quad (47)$$

Going ahead, let us note that Eq. (47) provides a rather good agreement of the theory and simulations for small particle velocities $v \ll v_{\rm res} = \omega_0/k_0$ only. The reason is that for resonant particles, the solution of the equation with the Fokker–Planck collision term does not provide proper description of the velocity dependence of D_ω inasmuch as β_ω is treated as a slowly dependent quantity. In order to take it into account more accurately, we should use a special expansion of D_ω over v which leads to a different form of the collision term. At the same time, it is possible to find an expression for the effective β_ω in terms of D_ω and then apply it in the calculations involving the Fokker–Planck collision term. This possibility is associated with the structure of the Fourier components of the transition probability and the diffusion coefficient D_ω. The idea is to express the exponent of the type $e^{-(k^2D/3)\tau^3}$ in terms of the effective collision frequency, namely,

$$e^{-(k^2D/3)\tau^3} \to e^{-\nu\tau}, \quad (48)$$

which is usually done in the quasilinear theory. However, in the traditional estimates the turbulent collision frequency ν is assumed to be determined by the turbulent diffusion only, i.e.,

$$\nu \simeq \left(\frac{k^2 D}{3}\right)^{1/3}. \tag{49}$$

This estimate follows from the assumption that

$$e^{-(k^2D/3)\tau^3} \simeq e^{-(k^2D\tau_0^2/3)\tau}$$

and $(k^2D\tau_0^3/3) \sim 1$.

Actually, such approximation does not provide in all cases the necessary accuracy for the integrals in terms of which the diffusion coefficients and mean-square displacements are determined.

In order to introduce an improved approximation let us define the turbulent collision frequency in terms of the equation,

$$\int_0^\infty d\tau e^{-(\Delta k^2 v^2 \tau^2/4) - (k_0^2 D_\omega/3)\tau^3} = \int_0^\infty d\tau e^{-(\Delta k^2 v^2\tau^2/4) - \nu_\omega \tau}. \tag{50}$$

Taking into account that

$$\int_0^\infty d\tau e^{-(\Delta k^2 v^2 \tau^2/4) - \nu_\omega \tau} = \frac{\sqrt{\pi}}{\Delta k v} e^{(\nu_\omega^2/\Delta k^2 v^2)}\left[1 - \mathrm{erf}\frac{\nu_\omega}{\Delta k v}\right],$$

Eq. (50) in the dimensionless form is given by

$$e^{x^2\xi^{2/3}/4}\left(1 - \mathrm{erf}\frac{x\xi^{1/3}}{2}\right) = \frac{2}{\sqrt{\pi}}\int_0^\infty dt e^{-t^2 - \xi t^3}, \tag{51}$$

where

$$x = \frac{\nu_\omega}{\left(k_0^2 \frac{D_\omega}{3}\right)^{1/3}}; \quad \xi = \frac{k_0^2 D_\omega/3}{(\Delta k v/2)^3}. \tag{52}$$

In the case of small ξ,

$$x(\xi) \simeq \xi^{2/3} \simeq \frac{4(k_0^2 D_\omega/3)^{2/3}}{\Delta k^2 v^2}. \tag{53}$$

For large ξ,

$$x(\xi) = \frac{3}{\Gamma(\frac{1}{3})}$$

and thus

$$\nu_\omega \simeq \frac{3}{\Gamma(\frac{1}{3})}\left(\frac{k_0^2 D_\omega}{3}\right)^{1/3},$$

which is of the same order as that estimated by Eq. (49).

In the general case we have

$$\nu_\omega(\xi) = \left(\frac{k_0^2 D_\omega}{3}\right)^{1/3} x(\xi), \tag{54}$$

where $x(\xi)$ is the solution of Eq. (51), and D_ω is given by Eq. (45). The result of the numerical tabulation of $x(\xi)$ is given in Fig. 1.

Within the approximation determined by Eq. (50), the mean-square velocity displacement can be written as

$$\begin{aligned}\langle \Delta v^2\rangle_\tau \simeq & -\frac{1}{v^2}\left(\frac{e}{m}\right)^2 \delta\Phi_0^2 \sum_{s=\pm 1}\int_0^\tau d\tau_1 \\ & \times \frac{1}{2\beta_0}(1 - e^{-2\beta_0(\tau-\tau_1)})\exp[-is\omega_0\tau_1 - \nu_s\tau_1]\frac{\partial^2}{\partial\tau_1^2} \\ & \times \exp\left[-\frac{\Delta k^2 v^2 \tau_1^2}{4} + isk_0 v\tau_1\right],\end{aligned} \tag{55}$$

where

$$\nu_s = \left(\frac{k_0^2 D_{-s\omega_0}}{3}\right)^{1/3} x(\xi). \tag{56}$$

After being integrated, Eq. (55) reduces to

$$\begin{aligned}\langle \Delta v^2\rangle_\tau = & \left(\frac{e}{m}\right)^2 \frac{\delta\Phi_0^2}{2v^2\beta_0}\sum_{s=\pm1}\left\{\omega_0^2 e^{-(\Omega+is\nu_0)^2/4\alpha}\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\left[\mathrm{erf}\sqrt{\alpha}\left(\tau + \frac{is\Omega+\nu_s}{2\alpha}\right) - \mathrm{erf}\frac{is\Omega+\nu_s}{2\sqrt{\alpha}}\right]\right. \\ & -(\omega_0 + is(2\beta_0+\nu_s))^2\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}e^{-(\Omega+is(2\beta_0-\nu_s))^2/4\alpha}e^{-2\beta_0\tau}\left[\mathrm{erf}\sqrt{\alpha}\left(\tau + \frac{is\Omega - 2\beta_0+\nu_s}{2\alpha}\right)\right. \\ & \left.\left. -\mathrm{erf}\frac{is\Omega-2\beta_0+\nu_s}{2\sqrt{\alpha}}\right] - 2\beta_0(e^{-\alpha\tau^2 - is\Omega\tau - \nu_s\tau} - 1)\right\},\end{aligned} \tag{57}$$

where

$$\alpha = \frac{\Delta k^2 v^2}{4}, \quad \Omega = \omega_0 - k_0 v.$$

Concerning the quantity $\beta_0 \equiv \beta_\omega|_{\omega\to 0}$, it is reasonable to estimate it using the approximation introduced above, i.e.,

$$\begin{aligned}\beta_0 &= \left(\frac{k_0^2 D_0}{3}\right)^{1/3} x(\xi), \\ \xi &= \frac{k_0^2 D_0}{3} \Big/ \left(\frac{\Delta k v}{2}\right)^3,\end{aligned} \tag{58}$$

where D_0 is the diffusion coefficient in the zeroth order approximation described by Eq. (45) for $\omega = 0$.

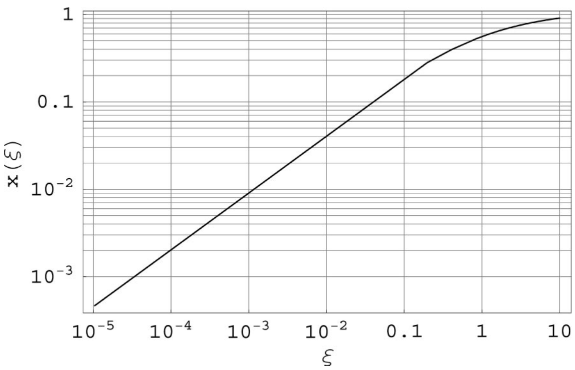

FIG. 1. The function $x(\xi)$ giving the solution of Eq. (51).

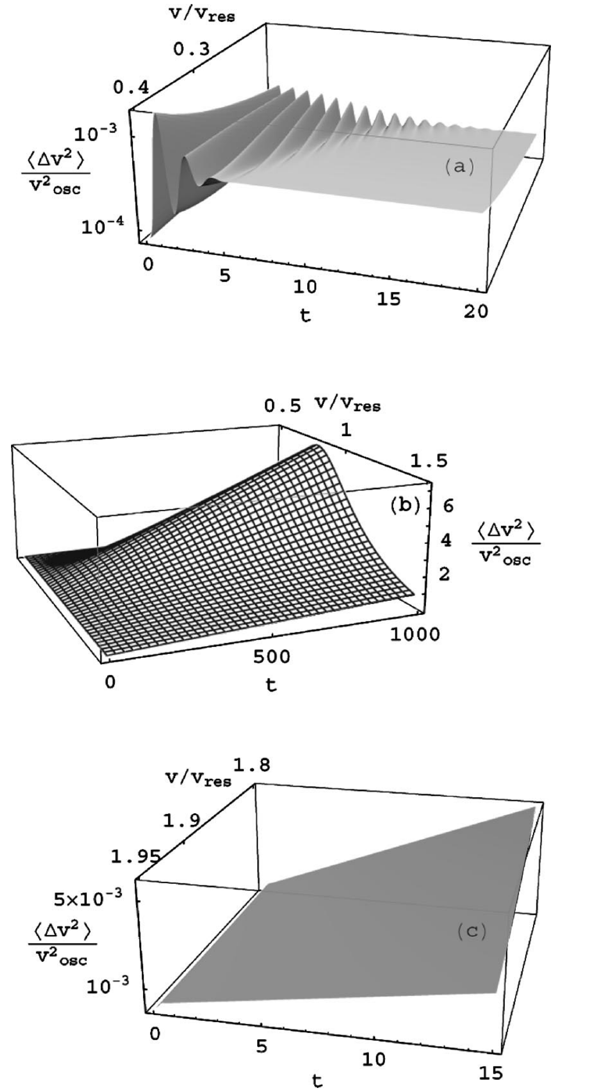

FIG. 2. The dependencies of the normalized mean-square velocity displacement $\langle \Delta v^2 \rangle / v^2_{\mathrm{osc}}$ on the dimensionless time $t = 2\pi\tau/\omega_0$ and velocity v/v_{res} in the case of weak fields ($\sigma = 10^{-4}$) and wide k-spectrum ($\delta k = 0.4$) for different velocity ranges.

If the fields are weak, then the results obtained are considerably simplified and reduce to the time-nonlocal quasilinear theory. In this case we can take $\nu_\omega = 0$ that leads to the Fourier-component of the diffusion coefficient described by Eq. (45) and

$$D(\tau,v) = \frac{1}{2}\left(\frac{e}{m}\right)^2 k_0^2 \delta\Phi_0^2 \sum_{s=\pm 1} e^{-(\Delta k^2 v^2 \tau^2/4) - is\Omega\tau} \times \left[\left(1 + i\frac{\Delta k^2}{2k_0^2} s k_0 v \tau\right)^2 + \frac{\Delta k^2}{4k_0^2}\right], \quad (59)$$

$$\langle \Delta v^2 \rangle_\tau = \left(\frac{e}{m}\right)^2 \delta\Phi^2 k_0^2 \frac{\sqrt{\pi}}{\Delta k v} \sum_{s=\pm 1} \left\{ e^{-\Omega^2/\Delta k^2 v^2} \left[\operatorname{erf}\sqrt{\alpha}\left(\tau + \frac{is\Omega}{2\alpha}\right) - \operatorname{erf}\frac{is\Omega}{2\sqrt{\alpha}}\right] \left[\tau + \frac{2is}{\Delta k v}\left(\frac{\Omega}{\Delta k v} - \frac{\Delta k v}{\omega_0}\right)\right] - \frac{2}{\sqrt{\pi}} \frac{1}{\Delta k v}\left(1 - \frac{\Delta k^2 v^2}{2\omega_0^2}\right) \times \left(1 - e^{-(\Delta k^2 v^2 \tau^2/4) + is\Omega\tau}\right) \right\}. \quad (60)$$

For $\Delta k v \tau_{\mathrm{ph}} \gg 1$ and $\Delta k v \tau_{\mathrm{ph}} \gg 4\Omega/\Delta k v$, where τ_{ph} is the physically infinitesimal time with respect to which the distribution function $f(X,t)$ [and thus the transition probability $W(X,X';t,t')$] are introduced, $D(\tau)$ could be approximated as

$$D(\tau) \simeq \left(\frac{e}{m}\right)^2 \delta\Phi_0^2 k_0^2 \frac{\sqrt{\pi}}{\Delta k v} \delta(\tau), \quad (61)$$

i.e., we are led to the Markovian limit. In this case we have

$$\langle \Delta v^2 \rangle_\tau \simeq 2\left(\frac{e}{m}\right)^2 \delta\Phi_0^2 k_0^2 \frac{\sqrt{\pi}}{\Delta k v} \tau, \quad (62)$$

$$\langle \Delta r^2 \rangle_\tau \simeq \frac{1}{3}\left(\frac{e}{m}\right)^2 \delta\Phi_0^2 k_0^2 \frac{\sqrt{\pi}}{\Delta k v} \tau^3. \quad (63)$$

If $\Delta k v \tau_{\mathrm{ph}} \ll 4\Omega/\Delta k v$, then the time-local approximation is no longer valid. In this case Eq. (60) reduces to

$$\langle \Delta v^2 \rangle_\tau \simeq 2\left(\frac{e}{m}\right)^2 \frac{k_0^2 \delta\Phi_0^2}{\Omega^2}\left(1 - e^{-\Delta k^2 v^2 \tau^2/4} \cos\Omega\tau\right), \quad (64)$$

that could be an illustration of the non-Markovian effect.

The time-nonlocality is also pronounced for resonant particles $v = v_{\mathrm{res}}$ (where $v_{\mathrm{res}} = \omega_0/k_0$) and $\Omega = 0$, if $\Delta k v \tau \leqslant 1$. In such a case, Eq. (60) yields

$$\langle \Delta v^2 \rangle_\tau \simeq \left(\frac{e}{m}\right)^2 k_0^2 \delta\Phi_0^2 \tau^2. \quad (65)$$

The dependencies of the mean-square velocity displacement for various values of $\Delta k/k_0$ and $\delta\Phi_0^2$ calculated on the basis of Eq. (57) are shown in Figs. 2–4. These figures illustrate the crucial role of the mentioned parameters on particle diffusion in the velocity space. For example, in the case of wide spectrum ($\delta k = \Delta k/k_0 = 0.4$) and weak field $\sigma \sim 10^{-4}$ [where $\sigma = (v_{\mathrm{osc}}/v_{\mathrm{res}})^2$, $v^2_{\mathrm{osc}} = (e/m)\delta\Phi_0$ and $v_{\mathrm{res}} = \omega_0/k_0$] diffusion of the resonant particles ($v = v_{\mathrm{res}}$) in the most cases

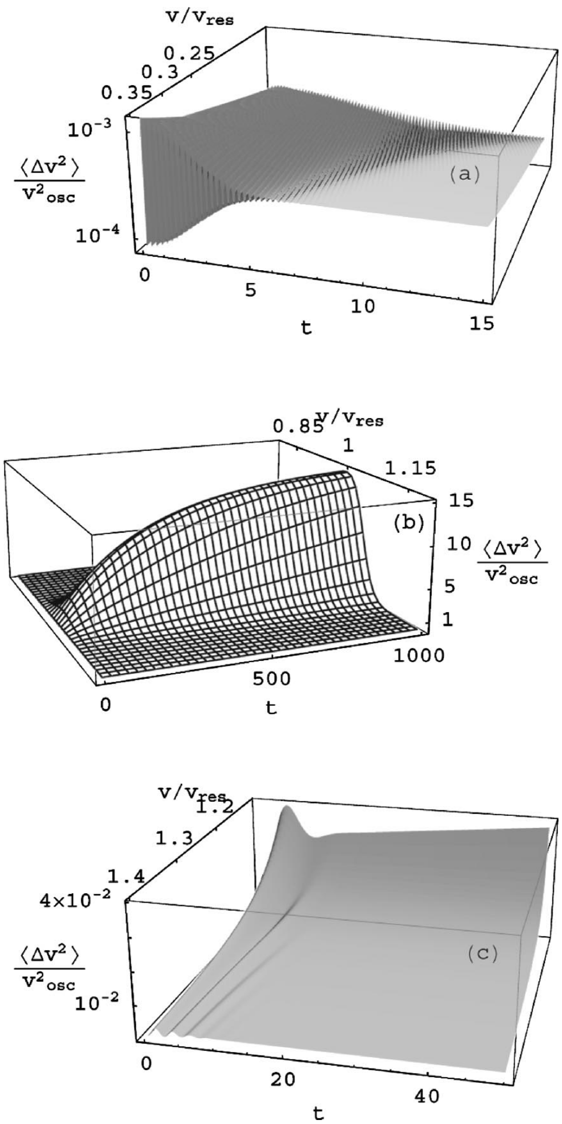

FIG. 3. The same dependencies as in Fig. 2 in the case of narrow k-spectrum ($\delta k=0.04$).

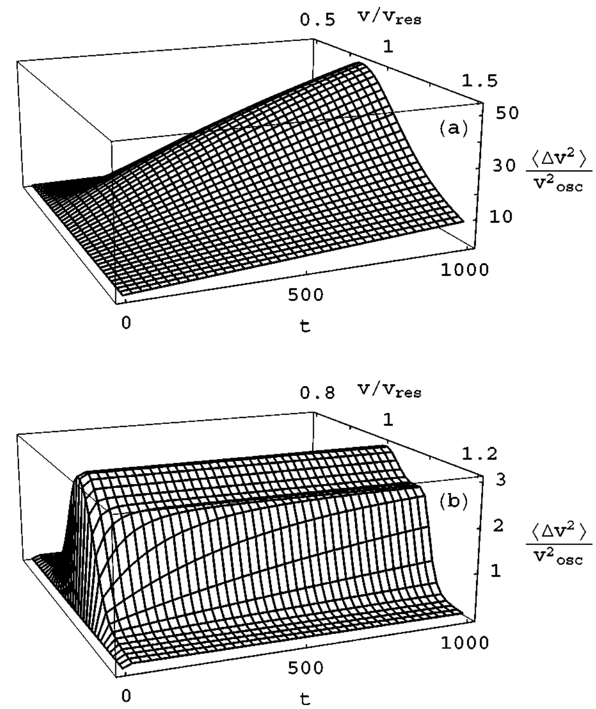

FIG. 4. The same dependencies as in Fig. 2 in the case of moderate fields ($\sigma=10^{-3}$) and different spectral width [(a) $\delta k=0.4$; (b) $\delta k=0.04$)] for particles with the velocities around $v_{\rm res}$. The normalized mean-square velocity displacement for strongly nonresonant particles coincides with that for the case $\sigma=10^{-4}$ [Figs. 2(a), 2(c), 3(b), 3(c)].

is described by time-local quasilinear theory, i.e., mean-square velocity displacement is proportional to the time interval τ [Fig. 2(b)]. The deviation from the time-local quasilinear theory is observed only at the initial stage of the process $\tau<\tau_{\rm cor}\sim 10$.

At the same time, for particles with velocity considerably deviated from the resonant velocity mean-square velocity displacement manifests time oscillations which is a typical time-nonlocal effect [Figs. 2(a), 3(a), 3(c)]. For the mentioned values of parameters mean-square velocity displacement is described by Eq. (60). However, the picture is changed with the decrease of the spectral width δk. As is seen [Fig. 3(b)] at $\delta k=0.04$ the Markovian quasilinear theory in the approximation of constant diffusion coefficient is more or less valid for moderately nonresonant particles but cannot be used anymore in the resonant case for which mean-square velocity displacement shows the tendency to saturation. The similar behavior is observed for the larger values of the intensity $\sigma=10^{-3}$ (Fig. 4). Concerning strongly nonresonant particles their mean-square velocity displacements in the case $\sigma=10^{-3}$, reproduce those for weak fields [Figs. 2(a), 2(c), 3(a), 3(c)]. The further decrease of δk leads to the fast saturation (for resonant particles), or frequent oscillation for even weakly nonresonant particles.

IV. NUMERICAL SIMULATIONS OF PARTICLE DIFFUSION IN RANDOM FIELDS

Particle diffusion in a random electric field was studied also numerically. The electric field was taken to be a superposition of N waves, i.e.,

$$\delta E=-\frac{\partial\delta\Phi}{\partial r},$$
$$\delta\Phi(r,t)=\sum_{i=1}^{N}\delta\Phi_i\cos(\omega_0 t-k_i r+\alpha_i), \tag{66}$$

where $\delta\Phi_i$ is the amplitude of the potential, k_i is the wave number, and α_i is the random phase of the ith mode. All waves have the same frequency ω_0. The minimum number of waves N was chosen so that for a typical run time interval and for negligibly small fields due to the discreteness of the spectrum the field correlation function would be recovered twice. But this does not occur for finite values of the amplitude taken in the simulations. The further increase of N did not show any noticeable effect on the results. Thus, the k-spectrum could be regarded as continuous.

The potential spectrum in the interval ($k_{\rm min}$, $k_{\rm max}$) was taken to be Gaussian, i.e.,

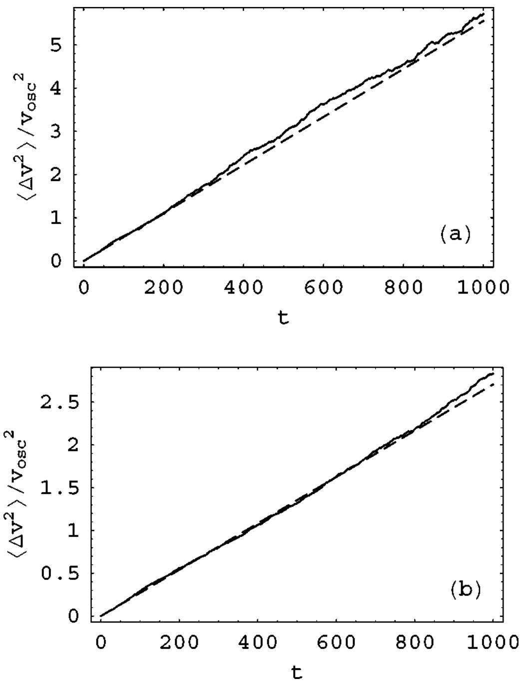

FIG. 5. The results of simulations of the mean-square velocity displacements in the case of weak field intensity ($\sigma = 10^{-4}$) and wide k-spectrum ($\delta k = 0.4$) for different particle initial velocities [(a) $v/v_{\rm res} = 1$; (b) $v/v_{\rm res} = 1.2$.] The dashed curves show appropriate theoretical dependencies.

$$\delta\Phi_i^2 = \delta\Phi_0^2 \frac{k_{\max} - k_{\min}}{\sqrt{\pi} N \Delta k} \exp\left[-\left(\frac{k_i - k_0}{\Delta k} \right)^2 \right], \quad i = 1, \ldots, N, \tag{67}$$

where k_0 corresponds to the central harmonic, Δk is the width of the spectrum, $\delta\Phi_0$ is the total field intensity. In the limit of continuous spectrum, $\delta\Phi_i \to \delta\Phi_k \, dk$,

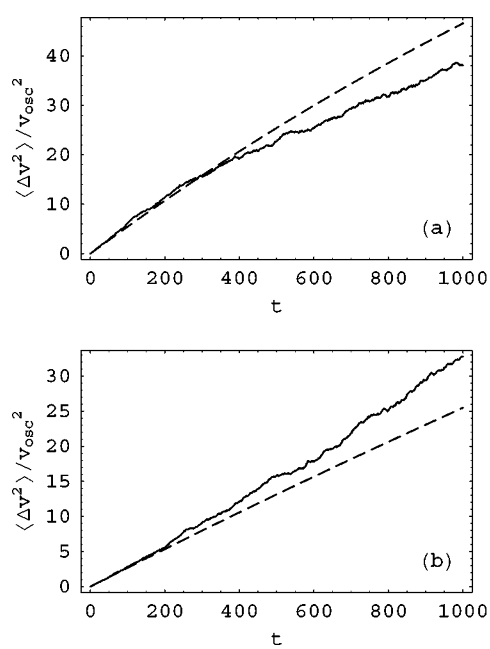

FIG. 6. The same dependencies as in Fig. 5 in the case of moderate field intensity ($\sigma = 10^{-3}$) and wide k-spectrum ($\delta k = 0.4$).

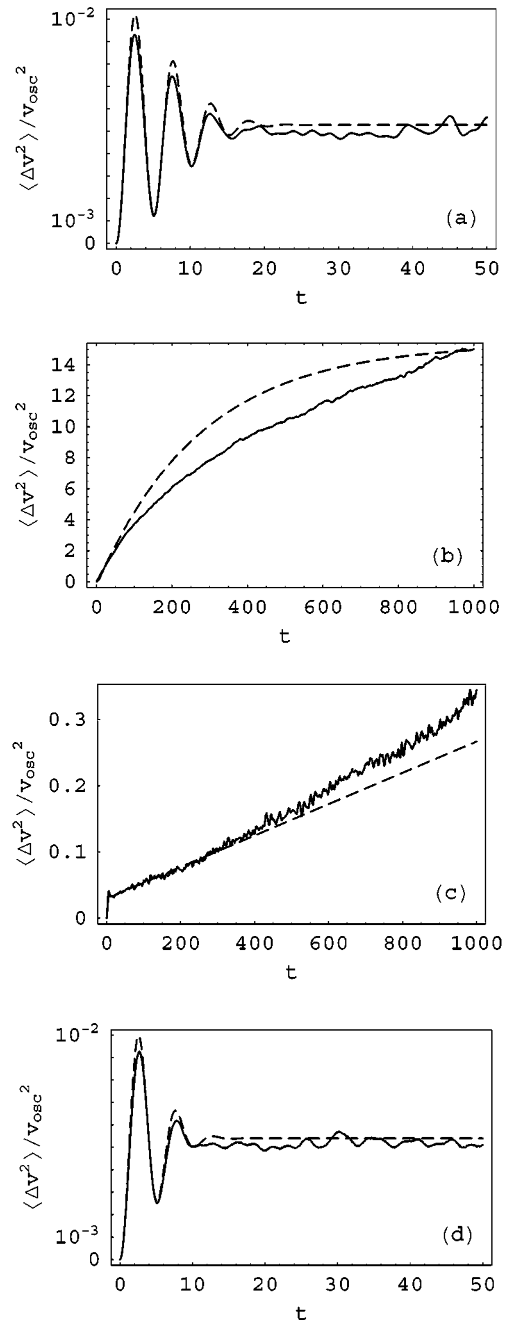

FIG. 7. The results of numerical simulations of the mean-square velocity displacements in the case of weak field intensity ($\sigma = 10^{-4}$) and narrow k-spectrum ($\delta k = 0.04$) for different particle initial velocities [(a) $v/v_{\rm res} = 0.8$; (b) $v/v_{\rm res} = 1$; (c) $v/v_{\rm res} = 1.1$; (d) $v/v_{\rm res} = 1.2$]. The dashed curves show appropriate theoretical dependencies.

$$\sum_i^N \delta\Phi_i^2 \to \int dk \, \delta\Phi_k^2 = \delta\Phi_0^2 .$$

The main characteristics of the random fields are their correlation functions. For the field (66) with the random phases α, all these are expressed in term of the pair correlation function, i.e.,

$$\langle \delta\Phi(r,t) \, \delta\Phi(r',t') \rangle = \delta\Phi_0^2 e^{-((\Delta k (r - r'))^2/4} \times \cos(k_0 (r - r') - \omega_0 (t - t')), \tag{68}$$

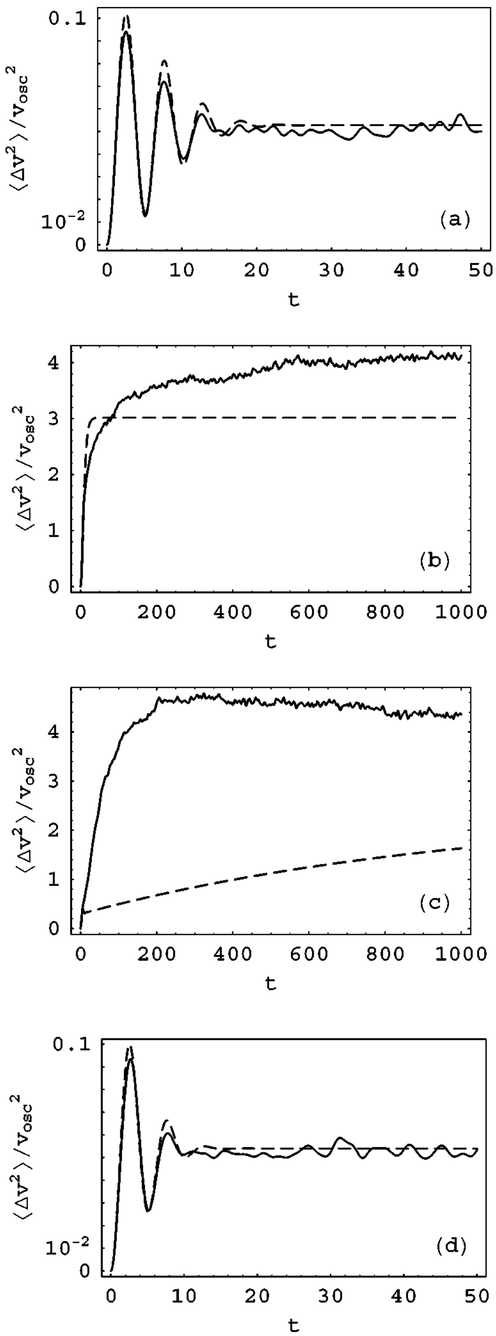

FIG. 8. The same dependencies as in Fig. 7 in the case of of moderate field intensity ($\sigma=10^{-3}$) and narrow k-spectrum ($\delta k=0.04$).

which is the same as one given by Eq. (33). It is known, however, that the higher correlation functions of the coordinates r and velocities v of particles moving in such field are not reduced to the pair correlation function because the equation of motion is nonlinear, i.e.,

$$\frac{dr}{dt}=v,$$
$$\frac{dv}{dt}=\frac{e}{m}\,\delta E(r,t). \tag{69}$$

The particle trajectories $v_j(t)$, $r_j(t)$ governed by Eqs. (66) and (69) were calculated numerically for n_r realizations

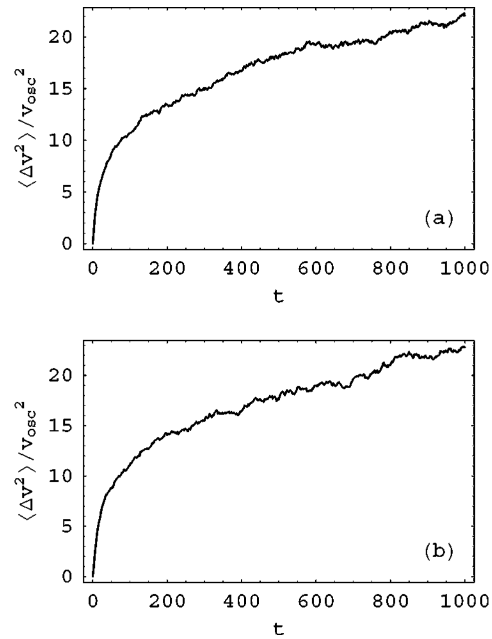

FIG. 9. The results of simulations of the mean-square velocity displacement in the case of strong fields ($\sigma=10^{-2}$) and wide spectrum ($\delta k=0.4$) for different particle initial velocities [(a) $v/v_{\mathrm{res}}=1$; (b) $v/v_{\mathrm{res}}=1.1$].

of the phases $\{\alpha_1,\ldots,\alpha_N\}_j$, $j=1,\ldots,n_r$. In each j realization, N phases $\{\alpha_1,\ldots,\alpha_N\}_j$ were given by the generator of random numbers from the uniform distribution over $(0,2\pi)$.

The average displacement in the velocity space $\overline{v}\equiv\langle v\rangle$ and the dispersion $\langle\Delta v^2\rangle_\tau=\langle(v-\overline{v})^2\rangle$ were found as a sum over n_r realizations,

$$\langle a(t)\rangle=\frac{1}{n_r}\sum_{j=1}^{n_r}a_j(t).$$

Note that the dispersion defined here is related to the mean-square deviation from the initial position a_0 as given by

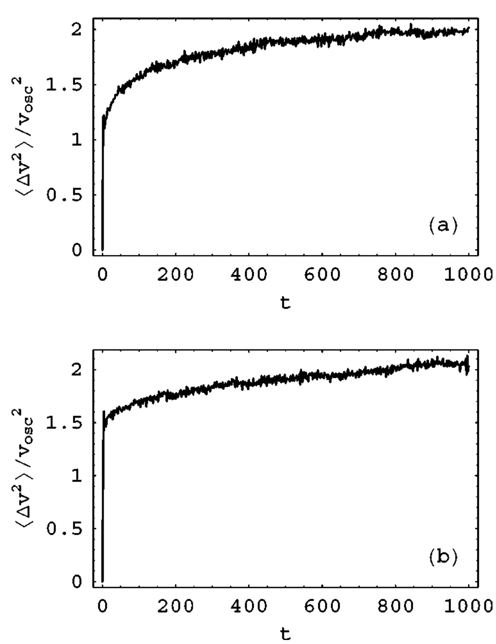

FIG. 10. The same dependencies as in Fig. 9 in the case of strong fields ($\sigma=10^{-2}$) and narrow spectrum ($\delta k=0.04$).

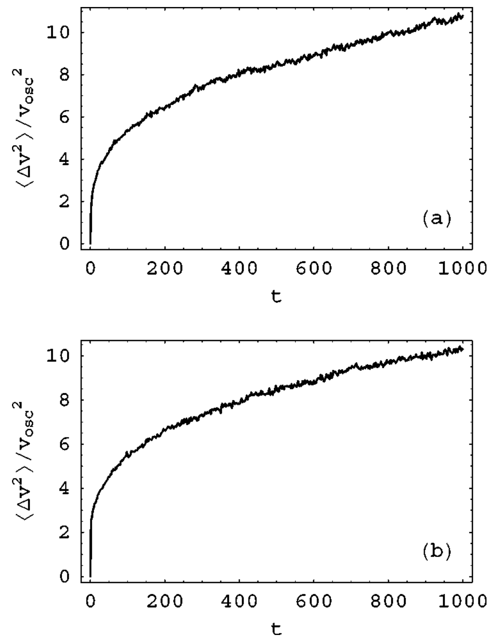

FIG. 11. The same dependencies as in Fig. 9 for $\sigma = 10^{-1}$ and $\delta k = 0.4$ [(a) $v/v_{\rm res} = 1$; (b) $v/v_{\rm res} = 1.5$].

$$\langle (a(t) - a_0)^2 \rangle = \langle (a(t) - \overline{a}(t))^2 \rangle + (\overline{a}(t) - a_0)^2.$$

We take the number of modes $N = 500-5000$ and the number of realizations $n_r = 500-4000$. For wide spectrum and weak field N should be greater then in other cases. The time interval in the units of the period $2\pi/\omega_0$ was taken to be equal to 1000. The unit of distance was the wave length of the central harmonic $2\pi/k_0$. The dimensionless potential $(e/m)\delta\Phi_0/v_{\rm res}^2$ is denoted by σ and $\Delta k/k_0$ by δk, $v_{\rm res} = \omega_0/k_0$.

We have considered both wide and narrow spectra, as well as weak and strong fields. The calculated results are shown in Figs. 5–11. As is seen from these figures the particle diffusion in the velocity space is considerably dependent on the parameters of turbulence in agreement with the predictions of time-nonlocal theory. The main features of the process under consideration are the following.

(i) In the case of weak field ($\sigma \leqslant 10^{-3}$) and rather wide spectrum ($\delta k \sim 0.4$) the mean-square velocity displacement of resonant (nearly resonant) particles is proportional to time interval (Figs. 5 and 6). The exception is the initial stage of the diffusion $\tau < \tau_{\rm cor}$, when $\langle \Delta v^2 \rangle_\tau$ is proportional to τ^2 (such small time scales are not shown in these figures).

(ii) Increase of the field intensity at the same spectral width leads to the saturation of the mean-square velocity displacements [compare Figs. 5, 6 and 9, 11 for the wide spectrum, Figs. 7(c) and 8(c), 10(b) for the narrow spectrum].

(iii) The value of the field intensity required for saturation of $\langle \Delta v^2 \rangle_\tau$ is dependent on the spectral width. The more narrow spectrum, the smaller field provides the saturation. For the wide spectrum Figs. 6(a), 6(b) do not show the saturation at $\sigma = 10^{-3}$, for the narrow spectrum Figs. 8(b), 8(c) show the saturation at this field intensity.

(iv) The saturation of $\langle \Delta v^2 \rangle_\tau$ is most easily achieved for resonant particles. Deviation of particle velocity from the resonant value leads to the increase of field intensity required for saturation [Fig. 7(b), saturation for the resonant particle at $\sigma = 10^{-4}$; and Fig. 8(c), saturation for nonresonant particles at higher field intensity].

(v) The evolution of mean-square velocity displacement is accompanied by frequent oscillations which become more pronounced with growing up of field intensity and narrowing of spectra (Figs. 7–10).

V. CONCLUSIONS

The simulations as well as the analytical estimates show the time behavior of the mean-square velocity displacement is crucially dependent on the width and the intensity of random field spectrum and particle initial velocity. Thus the diffusion of the resonant and nearly resonant particles is characterized by the linear time-dependence of $\langle \Delta v^2 \rangle_\tau$ only in the case of weak fields and wide k-spectrum, at $\tau > \tau_{\rm cor}$, that is the condition of the validity of the Markovian treatment. With field growing up and/or the spectrum narrowing the linear time-dependence gives place to mean-square velocity displacement saturation. The evolution of velocity dispersion is accompanied by the frequent oscillations which become more pronounced for strong fields and narrow spectra.

The proposed time-nonlocal theory of particle diffusion gives an appropriate description of the result of simulations, namely, the saturation of velocity dispersion for resonant particles and its oscillation for nonresonant ones in the case of relatively weak fields ($\sigma < 10^{-3}$). The extension of the theory for stronger fields would require more consistent treatment of the kinetic coefficients with due regard to their velocity dependence and time-nonlocal effects.

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