Electromagnetic formulation of global gyrokinetic particle simulation in toroidal geometry

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The fluid-kinetic hybrid electron model for global electromagnetic gyrokinetic particle simulations has been formulated in toroidal geometry using magnetic coordinates, providing the capabilities to describe low frequency processes in electromagnetic turbulence with electron dynamics. In the limit of long wavelength and no parallel electric field our equations reduce to the ideal magnetohydrodynamic equations. The formulation has been generalized to include equilibrium flows. The equations for zonal components of electrostatic and vector potentials have been derived, demonstrating the electron screening of the zonal vector potential. © *2009 American Institute of Physics*. [doi:10.1063/1.3273070]

I. INTRODUCTION

Gyrokinetic particle simulation has emerged as a powerful and reliable tool for describing plasma turbulence and transport since the first working code was demonstrated in 1983.[1] The advantage of the particle-in-cell (PIC) codes is their ability to simulate systems with a large number of degrees of freedom and their efficient use of the state-of-the-art computational resources.[2] Three dimensional electrostatic PIC simulations provide significant contribution to understanding drift wave turbulence such as ion temperature gradient (ITG),[2–9] electron temperature gradient,[10,11] and trapped electron mode (TEM)[12–14] turbulence. Recently, gyrokinetic particle simulations using the GTC code[2] have been applied to study the momentum transport[15] and energetic particle transport driven by the ITG turbulence.[16] Furthermore, in the presence of magnetic perturbations, there are other modes which play important roles in the turbulent transport, such as Alfvénic ion temperature gradient mode,[17,18] toroidal Alfvén eigenmode (TAE),[19] energetic particle mode (EPM),[20] kinetic ballooning mode,[21] etc. Electromagnetic simulations[22,23] are computationally more challenging, compared to the electrostatic simulations, due to the required resolution of the fast electron dynamics and the calculation of higher velocity moments of the particle distribution function. The purpose of this paper is to formulate a model which provides gyrokinetic particle simulation the capabilities to describe low frequency electromagnetic processes in toroidally confined plasmas, which can efficiently resolve above-mentioned computational challenges.

To justify the use of gyrokinetic description, the following ordering is adopted:

$$\frac{\omega}{\Omega} \sim \frac{k_\parallel}{k_\perp} \sim \frac{\delta B}{B_0} \sim \frac{e\phi}{T} \sim \mathcal{O}(\epsilon), \tag{1}$$

where ω, $k_\parallel$, and $k_\perp$ are the typical frequency, parallel and perpendicular wave numbers of the instability of interest, Ω is the ion cyclotron frequency, δB and ϕ are the perturbed magnetic field and electrostatic potentials, respectively, B_0 is the equilibrium magnetic field, and T is the plasma temperature. The gyrokinetic equation is used for ions, while electrons are described by the drift-kinetic equation for the ion-scale turbulence (ITG, TEM, TAE, EPM, etc.).

To improve the numerical properties of simulations with kinetic electrons the fluid-kinetic hybrid electron model[24] is developed. The idea of this model is to expand electron response and perturbed electromagnetic fields based on a small electron-to-ion mass ratio. In the lowest order, the electrons are adiabatic and can be described by fluid equations. Thus the unwanted high frequency electron modes are removed and no electron Courant condition needs to be observed. In the higher orders, the nonadiabatic electron response is treated kinetically with all nonlinear effects preserved, similar to that of the split-weight scheme.[25,26] The difference here is that the fluid-kinetic hybrid electron model removes tearing mode in the formulation to avoid the well-known numerical difficulty associated with the tearing physics,[27] while the split-weight scheme is an exact formulation of the drift-kinetic equation and thus treats all physics, including the tearing mode. We note that the tearing mode has not been efficiently resolved by any electron model in the toroidal gyrokinetic particle simulation. Basic equations of the hybrid electron model in the toroidal geometry have been formulated in Ref. 12 for electrostatic simulations and in Refs. 23 and 28 for electromagnetic simulations. In this work, we extend the formulation by adding zonal components of electrostatic and vector potentials, as well as radial electric field and mean flows, to provide a complete formulation of the hybrid model in the toroidal geometry.

The structure of this paper is the following. In Sec. II, we provide the starting equations using the well-known basic equations of the nonlinear gyrokinetic theory. In Sec. III, the fluid-kinetic hybrid electron model is formulated. In Sec. IV, we cast the equations using the magnetic coordinates, suitable for the toroidal geometry. In Sec. V, the equations for the zonal potentials are derived. In Sec. VI, the generalization for including the equilibrium flows is considered. Section VII demonstrates the reduction of our gyrokinetic formulation to the ideal magnetohydrodynamic (MHD) limit, recovering the dispersion relation of the shear-Alfvén waves. Finally, the conclusions are in Sec. VIII. In Appendixes A–E we provide the normalized, dimensionless form of equations for implementation in the simulation codes.

II. GYROKINETIC EQUATIONS IN TOROIDAL GEOMETRY

In the gyrokinetic particle simulations, the plasma is treated as a set of computational particles interacting with each other through self-generated electromagnetic fields. The simulation algorithm has two major steps: first, the calculation of the fields based on a given particle distribution and second, following the particle trajectories in these fields. Field equations are discretized on the three-dimensional spatial grids which are chosen to replicate the geometry of the equilibrium magnetic field (field-aligned mesh). The particle equations of motion are formulated in the magnetic coordinates.

We begin with the gyrokinetic equation describing toroidal plasmas in the inhomogeneous magnetic field,[29] using the gyrocenter position $\mathbf{X}$, magnetic moment μ, and parallel velocity $v_\parallel$ as a set of independent variables in the five-dimensional phase space,

$$\frac{d}{dt}f_\alpha(\mathbf{X},\mu,v_\parallel,t) \equiv \left[\frac{\partial}{\partial t} + \dot{\mathbf{X}}\cdot\nabla + \dot{v}_\parallel\frac{\partial}{\partial v_\parallel} - \mathcal{C}_\alpha\right]f_\alpha = 0, \tag{2}$$

$$\dot{\mathbf{X}} = v_\parallel\frac{\mathbf{B}}{B_0} + \mathbf{v}_E + \mathbf{v}_d,$$
$$\dot{v}_\parallel = -\frac{1}{m_\alpha}\frac{\mathbf{B}^*}{B_0}\cdot(\mu\nabla B_0 + Z_\alpha\nabla\phi) - \frac{Z_\alpha}{m_\alpha c}\frac{\partial A_\parallel}{\partial t}. \tag{3}$$

Here, index $\alpha=e,i$ stands for the particle species (electron or ion), Z_α is the particle charge, and m_α is the particle mass. $\mathbf{B}_0\equiv B_0\mathbf{b}_0$ is the equilibrium magnetic field, $\mathbf{B}\equiv\mathbf{B}_0+\delta\mathbf{B}$, and

$$\mathbf{B}^* = \mathbf{B}_0^* + \delta\mathbf{B} = \mathbf{B}_0 + \frac{B_0 v_\parallel}{\Omega_\alpha}\nabla\times\mathbf{b}_0 + \delta\mathbf{B}.$$

Other terms in Eq. (3) are the $E\times B$ drift velocity,

$$\mathbf{v}_E = \frac{c\mathbf{b}_0\times\nabla\phi}{B_0},$$

and magnetic drift velocity,

$$\mathbf{v}_d = \mathbf{v}_c + \mathbf{v}_g,$$

where the magnetic curvature drift is

$$\mathbf{v}_c = \frac{v_\parallel^2}{\Omega_\alpha}\nabla\times\mathbf{b}_0$$

and the grad-B drift is

$$\mathbf{v}_g = \frac{\mu}{m_\alpha\Omega_\alpha}\mathbf{b}_0\times\nabla B_0.$$

In our description we exclude the compressional component of the magnetic field perturbation by assuming $\delta B_\parallel=0$, thus

$$\delta\mathbf{B} = \delta\mathbf{B}_\perp = \nabla\times\lambda\mathbf{B}_0, \tag{4}$$

with $\lambda=A_\parallel/B_0$.

The collision operator $\mathcal{C}_\alpha$ has been implemented in GTC and is included in Eq. (2) for generality,[30,31] although we will omit it in the following sections, considering collisionless plasmas.

The electrostatic ϕ and vector potential $A_\parallel$ in Eq. (3) are being gyroaveraged for ions or taken at the gyrocenter position for electrons. The electrostatic potential can be found using gyrokinetic Poisson's equation,[32] assuming a single dominant ion species,

$$\frac{4\pi Z_i^2 n_i}{T_i}(\phi-\tilde{\phi}) = 4\pi(Z_i n_i - e n_e). \tag{5}$$

The vector potential satisfies the gyrokinetic Ampére's law,

$$\nabla_\perp^2 A_\parallel = \frac{4\pi}{c}(e n_e u_{\parallel e} - Z_i n_i u_{\parallel i}). \tag{6}$$

The density and parallel velocity are defined as the fluid moments of the corresponding distribution functions,

$$n_\alpha = \int d\mathbf{v} f_\alpha,$$

$$n_\alpha u_{\parallel\alpha} = \int d\mathbf{v}\, v_\parallel f_\alpha,$$

where

$$\int d\mathbf{v} \equiv \frac{\pi B_0}{m}\int dv_\parallel d\mu.$$

So far, Eqs. (2)–(6) can be readily implemented in a nonperturbative (full-f) simulation.[33,34] On the other hand, a perturbative δf simulation method[35–37] is commonly implemented in many gyrokinetic particle codes in order to minimize the discrete particle noise[38,39] for a better performance with much less number of particles compared to the full-f simulations. The idea of this method is to simulate only the evolution of the perturbed part of the distribution function governed by the nonlinear gyrokinetic equations. Beside the evaluation of phase space trajectories, an additional dynamic equation is solved for the particle weight, representing the contribution of the individual particle to the perturbed distribution function. The obtained quantity is used to calculate the velocity moments, such as the perturbed density and current, needed for solving the Maxwell's equations for electromagnetic fields.

The distribution function can be decomposed into equilibrium and perturbed parts $f_\alpha=f_{0\alpha}+\delta f_\alpha$ with the equilibrium part satisfying the gyrokinetic equation,

$$\frac{\partial f_{0\alpha}}{\partial t} + (v_\parallel\mathbf{b}_0 + \mathbf{v}_d)\cdot\nabla f_{0\alpha} - \frac{\mu}{m_\alpha}\frac{\mathbf{B}_0^*}{B_0}\cdot\nabla B_0\frac{\partial f_{0\alpha}}{\partial v_\parallel} - \mathcal{C}_\alpha(f_{0\alpha}) = 0. \tag{7}$$

Taking into account Eq. (7) and defining the particle weight as $w_\alpha\equiv\delta f_\alpha/f_\alpha$, we can rewrite Eq. (2) as

$$\frac{dw_\alpha}{dt} = (1-w_\alpha)\left[-\left(v_\parallel\frac{\delta\mathbf{B}}{B_0}+\mathbf{v}_E\right)\cdot\frac{\nabla f_{0\alpha}}{f_{0\alpha}} + \left(\mu\frac{\delta\mathbf{B}}{B_0}\cdot\nabla B_0 + Z_\alpha\frac{\mathbf{B}^*}{B_0}\cdot\nabla\phi + \frac{Z_\alpha}{c}\frac{\partial A_\parallel}{\partial t}\right)\times\frac{1}{m_\alpha}\frac{1}{f_{0\alpha}}\frac{\partial f_{0\alpha}}{\partial v_\parallel}\right]. \tag{8}$$

The dynamic equation (8) (for the perturbative simulation) or Eq. (2) (for full-f simulation) together with the field equations (5) and (6) form the basic closed system of equations for the nonlinear gyrokinetic simulations.

III. FLUID-KINETIC HYBRID ELECTRON MODEL

The fluid-kinetic hybrid electron model[24] is developed to improve the numerical properties of simulations with kinetic electrons for the ion-scale turbulence. The idea of hybrid method is to expand electron response into a dominant adiabatic part and a high-order kinetic perturbation, based on the small electron-ion mass ratio. The lowest order adiabatic part is described by the fluid equations, which removes numerical difficulties associated with the tearing modes and the electron Courant condition. The higher order nonadiabatic response is treated kinetically with all the nonlinear kinetic effects preserved. We separate the perturbed potentials into the flux-surface averaged (zonal) $\langle\phi\rangle$, $\langle A_\parallel\rangle$ and nonzonal $\delta\phi$, $\delta A_\parallel$ parts,

$$\phi = \delta\phi + \langle\phi\rangle,$$

$$A_\parallel = \delta A_\parallel + \langle A_\parallel\rangle.$$

The parallel electric field is

$$E_\parallel = -\mathbf{b}_0\cdot\nabla\phi - \frac{1}{c}\frac{\partial A_\parallel}{\partial t} = \delta E_\parallel - \frac{1}{c}\frac{\partial\langle A_\parallel\rangle}{\partial t}, \tag{9}$$

where

$$\delta E_\parallel = -\mathbf{b}_0\cdot\nabla\delta\phi - \frac{1}{c}\frac{\partial\delta A_\parallel}{\partial t}.$$

The adiabatic electron response only includes the nonzonal components of potentials.[24]

A. Fluid equations for adiabatic electrons

Integrating Eq. (2) and keeping terms up to the first order in the perturbation, we get the continuity equation for the electron density,

$$\frac{\partial\delta n_e}{\partial t} + B_0\mathbf{b}_0\cdot\nabla\left(\frac{n_0\delta u_{\parallel e}}{B_0}\right) + B_0\mathbf{v}_E\cdot\nabla\left(\frac{n_0}{B_0}\right) - n_0(\mathbf{v}_* + \mathbf{v}_E)\cdot\frac{\nabla B_0}{B_0} = 0, \tag{10}$$

where

$$\mathbf{v}_* = \frac{1}{n_0 m_e\Omega_e}\mathbf{b}_0\times\nabla(\delta P_\parallel + \delta P_\perp),$$

$$\delta P_\perp = \int d\mathbf{v}\,\mu B_0\delta f,$$

$$\delta P_\parallel = \int d\mathbf{v}\,m v_\parallel^2\delta f,$$

$$n_0 = \int d\mathbf{v} f_0.$$

Here, we make an approximation of $\nabla\times\mathbf{B}_0=0$ for deriving the continuity equation, although the actual magnetic field configuration and guiding center motion is noncurl-free. In this section, the subscript "e" is omitted when unnecessary.

The electron parallel fluid velocity can be calculated by inverting the Ampére's law (6),

$$n_0 e\,\delta u_{\parallel e} = \frac{c}{4\pi}\nabla_\perp^2\delta A_\parallel + n_0 Z_i\delta u_{\parallel i}. \tag{11}$$

Here $\delta u_{\parallel i}$ is the perturbed ion parallel flow velocity, which is calculated using the ion distribution function.

We define an effective scalar potential ϕ_{eff} to represent the parallel electric field,

$$\delta E_\parallel = -\mathbf{b}_0\cdot\nabla\phi_{\mathrm{eff}}.$$

Note that the linear tearing mode is removed here, since $\delta E_\parallel(k_\parallel=0)=0$.

We then define an inductive potential $\phi_{\mathrm{ind}}=\phi_{\mathrm{eff}}-\delta\phi$, such that the vector potential can be found from

$$\frac{\partial\delta A_\parallel}{\partial t} = c\mathbf{b}_0\cdot\nabla\phi_{\mathrm{ind}}. \tag{12}$$

In order to calculate the effective potential ϕ_{eff} we expand Eq. (2) for the electrons, keeping the leading order of $\omega/k_\parallel v_\parallel$,

$$v_\parallel\mathbf{b}_0\cdot\nabla\delta f^{(0)} = -v_\parallel\frac{\delta\mathbf{B}}{B_0}\cdot\nabla f_0\big|_{v_\perp} + v_\parallel\frac{f_0 e}{T_e}\mathbf{b}_0\cdot\nabla\phi_{\mathrm{eff}}. \tag{13}$$

Equation (13) has the solution

$$\delta f^{(0)} = \frac{ef_0}{T_e}\phi_{\mathrm{eff}} + \left.\frac{\partial f_0}{\partial\psi_0}\right|_{v_\perp}\delta\psi + \left.\frac{\partial f_0}{\partial\alpha_0}\right|_{v_\perp}\delta\alpha \tag{14}$$

describing the adiabatic electron response, i.e., electrons are isothermal along the perturbed magnetic field line, with the Boltzmann response to scalar potential. In deriving Eq. (14), we assume an equilibrium Maxwellian distribution for the parallel velocity with no inhomogeneity along the field line $\mathbf{b}_0\cdot\nabla f_0=0$. The notation $\nabla f|_{v_\perp}$ means derivative taken at $v_\perp=$const instead of $\mu=$const. We have also used the Clebsch representation for the toroidal magnetic field,

$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B} = \nabla(\psi_0 + \delta\psi) \times \nabla(\alpha_0 + \delta\alpha), \tag{15}$$

where ψ is the poloidal flux label, and $\alpha = q(\psi)\theta - \zeta$ is the magnetic field line label with θ and ζ, respectively, being the poloidal and toroidal angles in magnetic coordinates.

Since the electron response is adiabatic only for nonzonal ($k_\parallel \neq 0$) modes, Eq. (13) excludes zonal components (with $k_\parallel = 0$) of the perturbed potentials. Calculation of zonal components is discussed in Sec. V.

To find ϕ_{eff} in the lowest order, we integrate Eq. (14) over the velocity space,

$$\frac{e\phi_{\text{eff}}^{(0)}}{T_e} = \frac{\delta n_e}{n_0} - \frac{\delta\psi}{n_0}\frac{\partial n_0}{\partial\psi_0} - \frac{\delta\alpha}{n_0}\frac{\partial n_0}{\partial\alpha_0}. \tag{16}$$

The magnetic field line perturbations $\delta\psi$ and $\delta\alpha$ can be found using Eqs. (4) and (15) and assuming $\nabla \times \mathbf{B}_0 = 0$,

$$\frac{\partial\lambda}{\partial\psi_0} = \frac{\mathbf{B}_0}{B_0^2}\cdot\nabla\delta\alpha,$$
$$\frac{\partial\lambda}{\partial\alpha_0} = -\frac{\mathbf{B}_0}{B_0^2}\cdot\nabla\delta\psi. \tag{17}$$

Combining Eq. (17) and Eq. (12) gives

$$\frac{\partial\delta\psi}{\partial t} = -c\frac{\partial\phi_{\text{ind}}}{\partial\alpha_0},$$
$$\frac{\partial\delta\alpha}{\partial t} = c\frac{\partial\phi_{\text{ind}}}{\partial\psi_0}. \tag{18}$$

The pressure terms in Eq. (10) are correspondingly decomposed into two parts: through the separation of the electron distribution function into a zero-order adiabatic response $\delta f^{(0)}$ and a higher order nonadiabatic response δh. Using the solution Eq. (14) we can write

$$\delta P_\perp^{(0)} = \int d\mathbf{v}\mu B_0 \delta f^{(0)} = n_0 e\phi_{\text{eff}}^{(0)} + \frac{\partial(n_0T_e)}{\partial\psi_0}\delta\psi + \frac{\partial(n_0T_e)}{\partial\alpha_0}\delta\alpha, \tag{19}$$

$$\delta P_\parallel^{(0)} = \int d\mathbf{v} m v_\parallel^2 \delta f^{(0)} = n_0 e\phi_{\text{eff}}^{(0)} + \frac{\partial(n_0T_e)}{\partial\psi_0}\delta\psi + \frac{\partial(n_0T_e)}{\partial\alpha_0}\delta\alpha. \tag{20}$$

Note that the perturbed diamagnetic flow [fourth term in Eq. (10)] is smaller than the $E\times B$ convection [third term in Eq. (10)] by the factor $\phi_{\text{eff}}L_n/\delta\phi R_0 \ll 1$, and often can be neglected.

The obtained equations (10)–(12), (16), and (18)–(20), together with the gyrokinetic Poisson equation (5) and the ion contributions, form a closed nonlinear system in the lowest order of the electron-ion mass ratio expansion. These equations recover corresponding equations obtained in Refs. 23 and 28 but neglect the high-order nonlinear terms.

B. Kinetic equation for electron nonadiabatic response

The electron drift-kinetic equation (2) can be written as

$$Lf_e = 0,$$

where the propagator is separated into an equilibrium and perturbed parts $L = L_0 + \delta L$, with

$$L_0 = \frac{\partial}{\partial t} + (v_\parallel \mathbf{b}_0 + \mathbf{v}_d)\cdot\nabla - \frac{\mu}{m_e}\mathbf{b}_0\cdot\nabla B_0\frac{\partial}{\partial v_\parallel},$$

$$\delta L = \left(v_\parallel\frac{\delta\mathbf{B}}{B_0} + \mathbf{v}_E\right)\cdot\nabla - \left(\frac{\mu}{m_e}\frac{\delta\mathbf{B}}{B_0}\cdot\nabla B_0 + \frac{e}{m_e}E_\parallel - \frac{e}{m_e}\frac{\mathbf{v}_c}{v_\parallel}\cdot\nabla\phi\right)\frac{\partial}{\partial v_\parallel}.$$

The distribution function can be presented as a sum of

$$f_e = f_{0e} + \delta f_e = f_{0e} + \delta f_e^{(0)} + \delta h_e,$$

where f_{0e} is the equilibrium distribution, satisfying

$$L_0 f_{0e} = 0.$$

The perturbed distribution consists of the lowest order adiabatic solution $\delta f_e^{(0)}$ in Eq. (14) and a higher order kinetic correction δh_e.

The kinetic equation for δh_e can be written as

$$L\delta h_e = -\delta L f_{0e} - L_0\delta f_e^{(0)} - \langle\delta L\rangle\delta f_e^{(0)}, \tag{21}$$

by keeping the terms linear in perturbation and the zonal component.

Substituting $\delta f_e^{(0)}$ from Eq. (14) we get

$$L_0\delta f_e^{(0)} = f_{0e}L_0\left[\frac{e}{T_e}\phi_{\text{eff}}^{(0)} + \left.\frac{\partial\ln f_0}{\partial\psi_0}\right|_{v_\perp}\delta\psi + \left.\frac{\partial\ln f_0}{\partial\alpha_0}\right|_{v_\perp}\delta\alpha\right]$$
$$= f_{0e}\left[\frac{\partial}{\partial t} + (v_\parallel\mathbf{b}_0 + \mathbf{v}_d)\cdot\nabla\right] \times\left[\frac{e}{T_e}\phi_{\text{eff}}^{(0)} + \left.\frac{\partial\ln f_0}{\partial\psi_0}\right|_{v_\perp}\delta\psi + \left.\frac{\partial\ln f_0}{\partial\alpha_0}\right|_{v_\perp}\delta\alpha\right].$$

Taking into account the lowest order equation (13) and using Eq. (18), we can rewrite in terms of a particle weight for Eq. (21) (assuming $\delta f_e^{(0)}/f_0 \ll 1$ and $k_\perp L_p \sim 1/\epsilon$, where L_p is the equilibrium pressure inhomogeneity scale length),

$$\frac{1}{f_e}L\delta h_e = \frac{dw_e}{dt} = \left(1 - \frac{\delta f_e^{(0)}}{f_{0e}} - w_e\right)\left[-\mathbf{v}_E\cdot\nabla\ln f_{0e}|_{v_\perp} - \frac{\partial}{\partial t}\frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_d\cdot\nabla\frac{\delta f_e^{(0)}}{f_{0e}} + \frac{e}{T_e}\mathbf{v}_d\cdot\nabla\phi - \frac{c\mathbf{b}_0\times\nabla\langle\phi\rangle}{B_0}\cdot\nabla\frac{\delta f_e^{(0)}}{f_{0e}} + \frac{ev_\parallel}{cT_e}\frac{\partial\langle A_\parallel\rangle}{\partial t}\right], \tag{22}$$

where $w_e \equiv \delta h_e/f_e$,

$$\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{e}{T_e}\phi_{\text{eff}}^{(0)} + \left.\frac{\partial\ln f_{0e}}{\partial\psi_0}\right|_{v_\perp}\delta\psi + \left.\frac{\partial\ln f_{0e}}{\partial\alpha_0}\right|_{v_\perp}\delta\alpha,$$

and using Eq. (18)

$$\frac{\partial}{\partial t}\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{e}{T_e}\frac{\partial \phi_{\text{eff}}^{(0)}}{\partial t} + c\left.\frac{\partial \ln f_{0e}}{\partial \psi_0}\right|_{v_\perp}\frac{\partial \phi_{\text{ind}}}{\partial \alpha_0} + c\left.\frac{\partial \ln f_{0e}}{\partial \alpha_0}\right|_{v_\perp}\frac{\partial \phi_{\text{ind}}}{\partial \psi_0}.$$

The obtained equation (22) is equivalent to the one, derived in Refs. 23 and 28, except that we drop the high-order nonlinear terms and explicitly separate the electrostatic and vector potentials into zonal and nonzonal parts.

In order to find the first order correction $\phi_{\text{eff}}^{(1)}$ we integrate the perturbed distribution function using Eq. (14),

$$\frac{e\phi_{\text{eff}}^{(1)}}{T_e} = -\frac{\delta n_e^{(1)}}{n_{0e}}, \tag{23}$$

where

$$\delta n_e^{(1)} = \int d\mathbf{v}\,\delta h_e. \tag{24}$$

Equations (22)–(24) form a complete system for the first order correction of the nonadiabatic response. This procedure can be repeated to achieve higher accuracy. It was found that the second order accuracy is needed to recover the trapped electron response.[12]

IV. FORMULATION IN MAGNETIC COORDINATES

For tokamak simulations, it is convenient to use the toroidal magnetic coordinate system (ψ,θ,ζ), where ψ is a poloidal magnetic flux function, θ is a poloidal angle, and ζ is a toroidal angle. The covariant representation of the magnetic field in this system is

$$\mathbf{B}_0 = \delta\nabla\psi + I\nabla\theta + g\nabla\zeta, \tag{25}$$

the contravariant representation is

$$\mathbf{B}_0 = q\nabla\psi\times\nabla\theta - \nabla\psi\times\nabla\zeta, \tag{26}$$

and the Jacobian is

$$J^{-1} = \nabla\psi\cdot\nabla\theta\times\nabla\zeta = \frac{B_0^2}{gq+I}. \tag{27}$$

The radial component δ is small and usually neglected.[40]

The particle's equations of motion in the magnetic coordinates are[40]

$$\dot{\zeta} = \frac{v_\parallel B_0(q+\rho_c I' + I\partial_\psi \lambda)}{D} - c\frac{I}{D}\left[\frac{1}{Z_\alpha}\frac{\partial\varepsilon}{\partial B_0}\frac{\partial B_0}{\partial\psi} + \frac{\partial\phi}{\partial\psi}\right], \tag{28}$$

$$\dot{\theta} = \frac{v_\parallel B_0(1-\rho_c g' - g\partial_\psi \lambda)}{D} + c\frac{g}{D}\left[\frac{1}{Z_\alpha}\frac{\partial\varepsilon}{\partial B_0}\frac{\partial B_0}{\partial\psi} + \frac{\partial\phi}{\partial\psi}\right], \tag{29}$$

$$\dot{\psi} = \frac{c}{Z_\alpha}\frac{\partial\varepsilon}{\partial B_0}\left(\frac{I}{D}\frac{\partial B_0}{\partial\zeta} - \frac{g}{D}\frac{\partial B_0}{\partial\theta}\right) + \frac{cI}{D}\frac{\partial\phi}{\partial\zeta} - \frac{cg}{D}\frac{\partial\phi}{\partial\theta} + v_\parallel B_0\left(\frac{g}{D}\frac{\partial\lambda}{\partial\theta} - \frac{I}{D}\frac{\partial\lambda}{\partial\zeta}\right), \tag{30}$$

$$\begin{aligned}\dot{\rho}_\parallel = &-c\frac{(1-\rho_c g' - g\partial_\psi\lambda)}{D}\left[\frac{1}{Z_\alpha}\frac{\partial\varepsilon}{\partial B_0}\frac{\partial B_0}{\partial\theta} + \frac{\partial\phi}{\partial\theta}\right] \\ &- c\frac{(q+\rho_c I' + I\partial_\psi\lambda)}{D}\left[\frac{1}{Z_\alpha}\frac{\partial\varepsilon}{\partial B_0}\frac{\partial B_0}{\partial\zeta} + \frac{\partial\phi}{\partial\zeta}\right] \\ &+ c\frac{(I\partial_\zeta\lambda - g\partial_\theta\lambda)}{D}\left[\frac{1}{Z_\alpha}\frac{\partial\varepsilon}{\partial B_0}\frac{\partial B_0}{\partial\psi} + \frac{\partial\phi}{\partial\psi}\right] - \frac{\partial\lambda}{\partial t},\end{aligned} \tag{31}$$

where

$$D = gq + I + \rho_c(gI' - Ig').$$

The radial derivatives of poloidal and toroidal currents are

$$I' \equiv \frac{dI}{d\psi}, \quad g' \equiv \frac{dg}{d\psi}.$$

The modified parallel canonical momentum is

$$\rho_c = \rho_\parallel + \lambda,$$

with

$$\rho_\parallel = \frac{v_\parallel}{\Omega_\alpha} \equiv \frac{m_\alpha c}{Z_\alpha B_0}v_\parallel.$$

For short notation we use

$$\frac{\partial\varepsilon}{\partial B_0} = \mu + \frac{Z_\alpha^2}{m_\alpha c^2}\rho_\parallel^2 B_0.$$

Assuming Maxwellian background for the parallel velocity and keeping terms up to the first order in the perturbations, the gyrokinetic equation for ions, Eq. (8), can be written as

$$\frac{dw_i}{dt} = (1-w_i)\left[-\frac{c}{B_0}\mathbf{b}_0\times\nabla(\phi - v_\parallel\lambda B_0)\cdot\left.\frac{\nabla f_{0i}}{f_{0i}}\right|_{v_\perp} + \frac{Z_i}{T_i}v_\parallel E_\parallel - \frac{Z_i}{T_i}\mathbf{v}_d\cdot\nabla\phi\right]. \tag{32}$$

Here Eq. (4) and the gyrokinetic ordering $k_\perp R_0 \gg 1$ have been used.

Assuming only radial dependence of the background density and temperature, the first term in the square brackets of Eq. (32) becomes

$$\frac{c}{B_0}\mathbf{b}_0\times\nabla(\phi - v_\parallel\lambda B_0)\cdot\left.\frac{\nabla f_{0i}}{f_{0i}}\right|_{v_\perp} = \frac{c}{B_0^2 J}\left[I\frac{\partial}{\partial\zeta} - g\frac{\partial}{\partial\theta}\right](\phi - v_\parallel\lambda B_0)\left.\frac{1}{f_{0i}}\frac{\partial f_{0i}}{\partial\psi}\right|_{v_\perp}. \tag{33}$$

The scalar product is

$$\begin{aligned}\mathbf{v}_d\cdot\nabla\phi &= \left(\frac{m_i v_\parallel^2}{B_0} + \mu\right)\frac{1}{Z_i}\frac{c}{B_0^2}\mathbf{B}_0\times\nabla\mathbf{B}_0\cdot\nabla\phi \\ &= \left(\frac{m_i v_\parallel^2}{B_0} + \mu\right)\frac{1}{Z_i}\frac{c}{B_0^2 J} \\ &\quad\times\left(g\frac{\partial B_0}{\partial\psi}\frac{\partial\phi}{\partial\theta} - I\frac{\partial B_0}{\partial\psi}\frac{\partial\phi}{\partial\zeta} - g\frac{\partial B_0}{\partial\theta}\frac{\partial\phi}{\partial\psi}\right),\end{aligned} \tag{34}$$

assuming $\nabla\times\mathbf{B}_0=0$ and axisymmetry of the magnetic field.

The nonzonal component of parallel electric field is

$$\delta E_\parallel = -\mathbf{b}_0\cdot\nabla\phi_{\rm eff} = -\frac{1}{B_0}\frac{1}{J}\left(\frac{\partial\phi_{\rm eff}}{\partial\theta}+q\frac{\partial\phi_{\rm eff}}{\partial\zeta}\right), \quad (35)$$

and the zonal component is

$$\langle E_\parallel\rangle = -\frac{1}{c}\frac{\partial\langle A_\parallel\rangle}{\partial t}. \quad (36)$$

The high-order electron drift-kinetic equation (22) reads

$$\begin{aligned}\frac{1}{f_e}L\delta h_e = \frac{dw_e}{dt} = \left(1-\frac{\delta f_e^{(0)}}{f_{0e}}-w_e\right)\Bigg[&-\mathbf{v}_E\cdot\left.\frac{\nabla f_{0e}}{f_{0e}}\right|_{v_\perp}-\frac{\partial}{\partial t}\frac{\delta f_e^{(0)}}{f_{0e}}\\ &-\mathbf{v}_d\cdot\nabla\left(\frac{e}{T_e}\phi_{\rm ind}^{(0)}+\left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial\psi_0}\right|_{v_\perp}\delta\psi\right)\\ &+\left(\frac{e}{T_e}\mathbf{v}_d+\frac{c}{B_0}\mathbf{b}_0\times\nabla\frac{\delta f_e^{(0)}}{f_{0e}}\right)\cdot\nabla\langle\phi\rangle\\ &+\frac{ev_\parallel}{cT_e}\frac{\partial\langle A_\parallel\rangle}{\partial t}\Bigg], \end{aligned}\quad (37)$$

with the adiabatic solution

$$\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{e}{T_e}\phi_{\rm eff}^{(0)}+\left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial\psi_0}\right|_{v_\perp}\delta\psi$$

and background inhomogeneity drive

$$\mathbf{v}_E\cdot\left.\frac{\nabla f_{0e}}{f_{0e}}\right|_{v_\perp} = \frac{c}{B_0^2J}\left[I\frac{\partial\phi}{\partial\zeta}-g\frac{\partial\phi}{\partial\theta}\right]\frac{1}{f_0}\left.\frac{\partial f_0}{\partial\psi}\right|_{v_\perp}.$$

Using Eq. (18), the time derivative becomes

$$\frac{\partial}{\partial t}\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{e}{T_e}\frac{\partial\phi_{\rm eff}^{(0)}}{\partial t}-c\left.\frac{1}{f_0}\frac{\partial f_0}{\partial\psi}\right|_{v_\perp}\frac{1}{q}\frac{\partial\phi_{\rm ind}^{(0)}}{\partial\theta}.$$

The inductive potential and field-line perturbation convected by magnetic drift is

$$\begin{aligned}&\mathbf{v}_d\cdot\nabla\left(\frac{e}{T_e}\phi_{\rm ind}^{(0)}+\left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial\psi_0}\right|_{v_\perp}\delta\psi\right)\\ &=\left(\frac{m_ev_\parallel^2}{B_0}+\mu\right)\frac{1}{e}\frac{c}{B_0^2J}\times\left[I\frac{\partial B_0}{\partial\psi}\frac{\partial}{\partial\zeta}+g\frac{\partial B_0}{\partial\theta}\frac{\partial}{\partial\psi}-g\frac{\partial B_0}{\partial\psi}\frac{\partial}{\partial\theta}\right]\\ &\quad\times\left(\frac{e}{T_e}\phi_{\rm ind}^{(0)}+\left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial\psi_0}\right|_{v_\perp}\delta\psi\right).\end{aligned}$$

Finally, the zonal flow convection terms are

$$\mathbf{v}_d\cdot\nabla\langle\phi\rangle = \left(\frac{m_ev_\parallel^2}{B_0}+\mu\right)\frac{1}{e}\frac{c}{B_0^2J}g\frac{\partial B_0}{\partial\theta}\frac{\partial\langle\phi\rangle}{\partial\psi}$$

and

$$\begin{aligned}&\mathbf{b}_0\times\nabla\frac{\delta f_e^{(0)}}{f_{0e}}\cdot\nabla\langle\phi\rangle\\ &=\frac{\partial\langle\phi\rangle}{\partial\psi}\frac{1}{B_0J}\left[I\frac{\partial}{\partial\zeta}-g\frac{\partial}{\partial\theta}\right]\\ &\quad\times\left(\frac{e}{T_e}\phi_{\rm eff}^{(0)}+\left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial\psi_0}\right|_{v_\perp}\delta\psi\right).\end{aligned}$$

V. ZONAL FIELDS

The adiabatic solution of the drift-kinetic equation (14) has no zonal components in it, since electron response is adiabatic only for the nonzonal components. The lowest order zonal response can be found from the equation

$$\frac{\partial\langle\delta f_e^{(0)}\rangle}{\partial t} = f_{0e}\frac{v_\parallel e}{cT_e}\frac{\partial\langle A_\parallel\rangle}{\partial t}, \quad (38)$$

giving

$$\langle\delta f_e^{(0)}\rangle = f_{0e}\frac{v_\parallel e}{cT_e}\langle A_\parallel\rangle, \quad (39)$$

where the flux-surface averaging is defined as

$$\langle A\rangle \equiv \frac{\int Jd\theta d\zeta A}{\int Jd\theta d\zeta}. \quad (40)$$

Thus, the Ampére's law for zonal field can be written

$$\langle\nabla_\perp^2A_\parallel\rangle-\frac{1}{\delta_e^2}\langle A_\parallel\rangle = \frac{4\pi n_0}{c}(e\langle\delta u_{\parallel e}^{(1)}\rangle - Z_i\langle\delta u_{\parallel i}\rangle), \quad (41)$$

where $n_0\delta u_{\parallel e}^{(1)}=\int d\mathbf{v}v_\parallel\delta h_e$ and $\delta_e=c/\omega_{pe}$ is the electron collisionless skin depth. As we can see from Eq. (41), zonal currents on scale length larger than δ_e are strongly reduced by the electron shielding;[41] thus $\nabla_\perp^2$ in Eq. (41) can be neglected for the ion scale turbulence since typically $\rho_i>\delta_e$. Note that there is no such screening for $k_\parallel\neq 0$ component of $A_\parallel$ since the electron response is dominantly adiabatic. Thus, the solution for the zonal field is

$$\langle A_\parallel\rangle = \frac{4\pi n_0c}{\omega_{pe}^2}(Z_i\langle\delta u_{\parallel i}\rangle - e\langle\delta u_{\parallel e}^{(1)}\rangle). \quad (42)$$

The zonal component of the electrostatic potential can be found from the gyrokinetic Poisson equation (using the Padé approximation),

$$\frac{n_iZ_i^2}{m_i\Omega_i^2}\nabla_\perp^2\langle\phi\rangle = (1-\rho_i^2\nabla_\perp^2)(e\langle\delta n_e\rangle - Z_i\langle\delta n_i\rangle). \quad (43)$$

Alternatively, the general solution of the gyrokinetic Poisson equation (5) can be averaged over flux surface to obtain the zonal electrostatic potential.

VI. EQUILIBRIUM FLOWS

In many physical problems, such as the toroidal momentum transport simulations,[15] the presence of background flows is required. Rotating plasma can be modeled by including parallel flow velocity in the shifted Maxwellian equilibrium distribution,

$$f_{0\alpha}=\frac{n_{0\alpha}}{(2\pi T_{0\alpha}/m_\alpha)^{3/2}}\exp\left[-\frac{2\mu B_0+m(v_\parallel-v_{\parallel 0})^2}{2T_{0\alpha}}\right]. \quad (44)$$

Substituting this distribution into the right-hand side of the gyrokinetic equation (8), we obtain

$$\begin{aligned}\frac{dw_\alpha}{dt}&=(1-w_\alpha)\Bigg[-\left(v_\parallel\frac{\delta\mathbf{B}}{B_0}+\mathbf{v}_E\right)\cdot\boldsymbol{\kappa}+\frac{v_{\parallel 0}\mu}{T_{0\alpha}}\frac{\delta\mathbf{B}}{B_0}\cdot\nabla B_0\\&\quad-\frac{Z_\alpha}{T_{0\alpha}}(v_\parallel-v_{\parallel 0})\left(\mathbf{b}_0\cdot\nabla\phi+\frac{1}{c}\frac{\partial A_\parallel}{\partial t}\right)\\&\quad-\frac{Z_\alpha}{T_{0\alpha}}\left(\mathbf{v}_g+\left(1-\frac{v_{\parallel 0}}{v_\parallel}\right)\mathbf{v}_c\right)\cdot\nabla\phi\Bigg],\end{aligned} \quad (45)$$

where

$$\begin{aligned}\boldsymbol{\kappa}&=\frac{\nabla n_{0\alpha}}{n_{0\alpha}}+\left[\left(\frac{\mu B_0}{T_{0\alpha}}+\frac{m_\alpha(v_\parallel-v_{\parallel 0})^2}{2T_{0\alpha}}\right)-\frac{3}{2}\right]\frac{\nabla T_{0\alpha}}{T_{0\alpha}}\\&\quad+\frac{m(v_\parallel-v_{\parallel 0})\nabla v_{\parallel 0}}{T_{0\alpha}}\end{aligned}$$

represents the background density, temperature, and parallel flow gradients.

The parallel velocity shift $v_{\parallel 0}$ is taken to be equal to the local equilibrium parallel flow velocity $u_\parallel$, which is calculated using the radial force balance equation and neoclassical collisionality,[42]

$$u_\parallel=-\frac{cgT_{0i}}{Z_iB_0}\left(\frac{Z_i}{T_{0i}}\frac{\partial\phi_0}{\partial\psi}+(1-k)\frac{\partial\ln T_{0i}}{\partial\psi}+\frac{\partial\ln n_{0i}}{\partial\psi}\right). \quad (46)$$

The corresponding neoclassical poloidal flow is

$$u_\theta=k\frac{cgT_{0i}B_{0\theta}}{Z_iB_0^2}\frac{\partial\ln T_{0i}}{\partial\psi}. \quad (47)$$

Here k is the poloidal rotation factor ($k\simeq 1.17$ in the banana regime), and ϕ_0 is the equilibrium electrostatic potential.

The high order drift kinetic equation for electrons, Eq. (22), in case of shifted Maxwellian background distribution becomes

$$\begin{aligned}\frac{dw_e}{dt}&=\left(1-\frac{\delta f_e^{(0)}}{f_{0e}}-w_e\right)\Bigg[-\mathbf{v}_E\cdot\boldsymbol{\kappa}-\frac{\partial}{\partial t}\frac{\delta f_e^{(0)}}{f_{0e}}-\mathbf{v}_d\cdot\nabla\frac{\delta f_e^{(0)}}{f_{0e}}\\&\quad+\left(\mathbf{v}_g+\mathbf{v}_c\left(1-\frac{v_{\parallel 0}}{v_\parallel}\right)\right)\cdot\nabla\phi-\frac{c\mathbf{b}_0\times\nabla\langle\phi\rangle}{B_0}\cdot\nabla\frac{\delta f_e^{(0)}}{f_{0e}}\\&\quad+(v_\parallel-v_{\parallel 0})\frac{e}{cT_e}\frac{\partial\langle A_\parallel\rangle}{\partial t}+\frac{e}{T_e}v_{\parallel 0}\mathbf{b}_0\cdot\nabla\phi_{\text{eff}}\\&\quad+v_{\parallel 0}\frac{\mu}{T_e}\frac{\delta\mathbf{B}}{B_0}\cdot\nabla B_0\Bigg].\end{aligned} \quad (48)$$

VII. REDUCTION TO IDEAL MHD LIMIT

In order to demonstrate that our simulation retains ideal MHD modes, such as TAE, EPM, etc., we reduce our equations in the limit of long wavelength and no parallel electric field to recover the well known MHD equations.[43]

In the ideal MHD $\phi_{\text{eff}}=0$. Assuming $\omega^*\ll\omega_A=k_\parallel v_A$, the continuity equation (10) can be written

$$\frac{\partial}{\partial t}\left(\frac{\delta n_e}{n_0}\right)+B_0\mathbf{b}_0\cdot\nabla\left(\frac{\delta u_{\parallel e}}{B_0}\right)-2\mathbf{v}_E\cdot\frac{\nabla B_0}{B_0}=0. \quad (49)$$

Note that the last term cancels out with the same contribution from ions.

The Poisson equation (5) in the long wavelength limit becomes[32]

$$\frac{c^2}{4\pi e}\nabla_\perp\left(\frac{1}{v_A^2}\nabla_\perp\phi\right)=\delta n_e. \quad (50)$$

Applying $\nabla_\perp^2$ operator on Eq. (12) gives

$$\frac{1}{c}\frac{\partial}{\partial t}(\nabla_\perp^2 A_\parallel)=-\mathbf{b}_0\cdot\nabla(\nabla_\perp^2\phi). \quad (51)$$

The inverse Ampére's law Eq. (11) reads

$$\delta u_{\parallel e}=c\frac{e}{T_e}\lambda_D^2(\nabla_\perp^2 A_\parallel). \quad (52)$$

The ion density and ion parallel velocity have been ignored in Eqs. (50) and (51), since the ion parallel current is much smaller than the electron parallel current.

Finally, the linear dispersion relation based on Eqs. (49)–(52) is

$$\omega^2\nabla_\perp\left(\frac{1}{v_A^2}\nabla_\perp\phi\right)=-B_0\mathbf{b}_0\cdot\nabla\left[\frac{1}{B_0}\nabla^2(\mathbf{b}_0\cdot\nabla\phi)\right], \quad (53)$$

which recovers the ideal MHD equations in Ref. 43 and similar derivations in Refs. 23 and 44.

VIII. CONCLUSIONS

The fluid-kinetic hybrid electron model for global electromagnetic gyrokinetic particle simulations has been formulated in the toroidal geometry using magnetic coordinates. The model provides capabilities to describe low frequency electromagnetic processes in the electromagnetic turbulence with the electron dynamics. In the limit of long wavelength and no parallel electric field our equations reduce to the well-known ideal MHD equations. The description has been generalized to include equilibrium flows. The equations for zonal components of electrostatic and vector potentials have been derived, demonstrating the electron screening of the zonal vector potential.

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APPENDIX A: UNITS AND NORMALIZATION

The basic units and normalization used in GTC are summarized in the appendix.

Basic units:

- time: inverse on-axis cyclotron frequency of proton $\Omega_p^{-1}=m_pc/eB_a$,

- length: tokamak major radius R_0,
- charge: proton charge e, and
- mass: proton mass m_p.

Dimensionless number: $\beta_e = 8\pi n_a T_e / B_0^2$.

Normalizations (B_a is the on-axis equilibrium magnetic field and n_a is the on-axis equilibrium electron density):

Magnetic field

$$\hat{B} = \frac{B}{B_a}.$$

Density

$$\hat{n} = \frac{n}{n_a}.$$

Scalar potential

$$\hat{\phi} = \frac{e}{m_p R_0^2 \Omega_p^2}\phi.$$

Vector potential

$$\hat{A} = \frac{A}{B_a R_0}.$$

Current

$$\hat{I} = \frac{I}{B_a R_0}.$$

Poloidal flux function (same for α)

$$\hat{\psi} = \frac{\psi}{B_a R_0^2}.$$

Velocity

$$\hat{v} = \frac{v}{R_0 \Omega_p}.$$

Temperature (energy)

$$\hat{T} = \frac{T}{m_p R_0^2 \Omega_p^2}.$$

Pressure

$$\hat{P} = \frac{P}{n_a m_p R_0^2 \Omega_p^2}.$$

In the following equations, the hats over dimensionless variables are omitted.

APPENDIX B: NORMALIZED ION EQUATIONS

Normalized Eq. (32),

$$\frac{dw_i}{dt} = (1 - w_i)\left[-\frac{1}{B_0}\mathbf{b}_0 \times \nabla(\phi - v_\parallel \lambda B_0) \cdot \left. \frac{\nabla f_0}{f_0} \right|_{v_\perp} + \frac{Z_i}{T_i} v_\parallel E_\parallel - \frac{Z_i}{T_i}\mathbf{v}_d \cdot \nabla \phi \right], \quad \text{(B1)}$$

where

$$\mathbf{b}_0 \times \nabla(\phi - v_\parallel \lambda B_0) \cdot \left. \frac{\nabla f_{0i}}{f_{0i}} \right|_{v_\perp} = \frac{1}{B_0 J}\left[I\frac{\partial}{\partial \zeta} - g\frac{\partial}{\partial \theta} \right](\phi - v_\parallel \lambda B_0) \left. \frac{1}{f_{0i}}\frac{\partial f_{0i}}{\partial \psi} \right|_{v_\perp},$$

$$E_\parallel = -\frac{1}{B_0}\frac{1}{J}\left(\frac{\partial \phi_{\text{eff}}}{\partial \theta} + q\frac{\partial \phi_{\text{eff}}}{\partial \zeta} \right) - \frac{\partial \langle A_\parallel \rangle}{\partial t},$$

$$\mathbf{v}_d \cdot \nabla \phi = \frac{\partial \varepsilon}{\partial B_0}\frac{1}{Z_i}\frac{1}{B_0^2 J}\left(g\frac{\partial B_0}{\partial \psi}\frac{\partial \phi}{\partial \theta} - I\frac{\partial B_0}{\partial \psi}\frac{\partial \phi}{\partial \zeta} - g\frac{\partial B_0}{\partial \theta}\frac{\partial \phi}{\partial \psi} \right).$$

APPENDIX C: NORMALIZED EQUATIONS OF MOTION

Equations of motion (28)–(31),

$$\dot{\zeta} = \frac{v_\parallel B_0 (q + \rho_c I' + I\partial_\psi \lambda)}{D} - \frac{I}{D}\left[\frac{1}{Z_\alpha}\frac{\partial \varepsilon}{\partial B_0}\frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right], \quad \text{(C1)}$$

$$\dot{\theta} = \frac{v_\parallel B_0 (1 - \rho_c g' - g\partial_\psi \lambda)}{D} + \frac{g}{D}\left[\frac{1}{Z_\alpha}\frac{\partial \varepsilon}{\partial B_0}\frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right], \quad \text{(C2)}$$

$$\dot{\psi} = \frac{1}{Z_\alpha}\frac{\partial \varepsilon}{\partial B_0}\left(\frac{I}{D}\frac{\partial B_0}{\partial \zeta} - \frac{g}{D}\frac{\partial B_0}{\partial \theta} \right) + \frac{I}{D}\frac{\partial \phi}{\partial \zeta} - \frac{g}{D}\frac{\partial \phi}{\partial \theta} + v_\parallel B_0 \left(\frac{g}{D}\frac{\partial \lambda}{\partial \theta} - \frac{I}{D}\frac{\partial \lambda}{\partial \zeta} \right), \quad \text{(C3)}$$

$$\dot{\rho}_\parallel = -\frac{(1 - \rho_c g' - g\partial_\psi \lambda)}{D}\left[\frac{1}{Z_\alpha}\frac{\partial \varepsilon}{\partial B_0}\frac{\partial B_0}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right] - \frac{(q + \rho_c I' + I\partial_\psi \lambda)}{D}\left[\frac{1}{Z_\alpha}\frac{\partial \varepsilon}{\partial B_0}\frac{\partial B_0}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} \right] + \frac{(I\partial_\zeta \lambda - g\partial_\theta \lambda)}{D}\left[\frac{1}{Z_\alpha}\frac{\partial \varepsilon}{\partial B_0}\frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right] - \frac{\partial \lambda}{\partial t}, \quad \text{(C4)}$$

where

$$D = gq + I + \rho_c(gI' - Ig'),$$

$$I' \equiv \frac{dI}{d\psi}, \quad g' \equiv \frac{dg}{d\psi},$$

$$\rho_c = \rho_\parallel + \lambda,$$

$$\rho_\parallel = \frac{v_\parallel}{\Omega_\alpha} \equiv \frac{m_\alpha}{Z_\alpha B_0}v_\parallel,$$

$$\frac{\partial \varepsilon}{\partial B_0} = \mu + \frac{Z_\alpha^2}{m_\alpha}\rho_\parallel^2 B_0.$$

APPENDIX D: NORMALIZED ELECTRON EQUATIONS

Dimensionless continuity equation (10),

$$\frac{\partial \delta n_e}{\partial t} + B_0 \mathbf{b}_0 \cdot \nabla\left(\frac{n_0 \delta u_{\parallel e}}{B_0}\right) + B_0 \mathbf{v}_E \cdot \nabla\left(\frac{n_0}{B_0}\right) - n_0(\mathbf{v}_* + \mathbf{v}_E)\cdot \frac{\nabla B_0}{B_0} = 0, \tag{D1}$$

where

$$\mathbf{v}_E = \frac{\mathbf{b}_0 \times \nabla \phi}{B_0},$$

$$\mathbf{v}_* = -\frac{1}{n_0 B_0}\mathbf{b}_0 \times \nabla(\delta P_\parallel + \delta P_\perp), \tag{D2}$$

$$\delta P_\perp^{(0)} = n_0 \phi_{\mathrm{eff}}^{(0)} + \frac{\partial(n_0 T_e)}{\partial \psi}\delta\psi,$$

$$\delta P_\parallel^{(0)} = n_0 \phi_{\mathrm{eff}}^{(0)} + \frac{\partial(n_0 T_e)}{\partial \psi}\delta\psi. \tag{D3}$$

Inverse Ampére's law Eq. (11),

$$\delta u_{\parallel e} = \frac{2T_{ea}}{\beta_{ea}}\frac{1}{n_0}\nabla_\perp^2 \delta A_\parallel + \frac{Z_i n_i}{n_0}\delta u_{\parallel i}, \tag{D4}$$

where β_{ea} is the on-axis value of β_e and T_{ea} is the normalized on-axis electron temperature.

High-order electron drift-kinetic equation (37),

$$\frac{dw_e}{dt} = \left(1 - \frac{\delta f_e^{(0)}}{f_{0e}} - w_e\right)\left[-\mathbf{v}_E \cdot \left.\frac{\nabla f_{0e}}{f_{0e}}\right|_{v_\perp} - \frac{\partial}{\partial t}\frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_d \cdot \nabla\left(\frac{\phi_{\mathrm{ind}}^{(0)}}{T_e} + \left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial \psi_0}\right|_{v_\perp}\delta\psi\right) + \left(\frac{\mathbf{v}_d}{T_e} + \frac{1}{B_0}\mathbf{b}_0 \times \nabla \frac{\delta f_e^{(0)}}{f_{0e}}\right)\cdot \nabla\langle\phi\rangle + \frac{v_\parallel}{T_e}\frac{\partial\langle A_\parallel\rangle}{\partial t}\right], \tag{D5}$$

with

$$\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{\phi_{\mathrm{eff}}^{(0)}}{T_e} + \left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial \psi_0}\right|_{v_\perp}\delta\psi,$$

$$\mathbf{v}_E \cdot \left.\frac{\nabla f_{0e}}{f_{0e}}\right|_{v_\perp} = \frac{1}{B_0^2 J}\left[I\frac{\partial \phi}{\partial \zeta} - g\frac{\partial \phi}{\partial \theta}\right]\frac{1}{f_0}\left.\frac{\partial f_0}{\partial \psi}\right|_{v_\perp},$$

$$\frac{\partial}{\partial t}\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{1}{T_e}\frac{\partial \phi_{\mathrm{eff}}^{(0)}}{\partial t} - \left.\frac{1}{f_0}\frac{\partial f_0}{\partial \psi}\right|_{v_\perp}\frac{1}{q}\frac{\partial \phi_{\mathrm{ind}}^{(0)}}{\partial \theta},$$

$$\mathbf{v}_d \cdot \nabla\langle\phi\rangle = \frac{\partial \varepsilon}{\partial B_0}\frac{1}{B_0^2 J}g\frac{\partial B_0}{\partial \theta}\frac{\partial\langle\phi\rangle}{\partial \psi},$$

$$\mathbf{b}_0 \times \nabla \frac{\delta f_e^{(0)}}{f_{0e}} \cdot \nabla\langle\phi\rangle = \frac{\partial\langle\phi\rangle}{\partial \psi}\frac{1}{B_0 J}\left[I\frac{\partial}{\partial \zeta} - g\frac{\partial}{\partial \theta}\right]\left(\frac{\phi_{\mathrm{eff}}^{(0)}}{T_e} + \left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial \psi_0}\right|_{v_\perp}\delta\psi\right),$$

$$\mathbf{v}_d \cdot \nabla\left(\frac{1}{T_e}\phi_{\mathrm{ind}}^{(0)} + \left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial \psi_0}\right|_{v_\perp}\delta\psi\right) = \frac{\partial \varepsilon}{\partial B_0}\frac{1}{B_0^2 J}\times\left[I\frac{\partial B_0}{\partial \psi}\frac{\partial}{\partial \zeta} + g\frac{\partial B_0}{\partial \theta}\frac{\partial}{\partial \psi} - g\frac{\partial B_0}{\partial \psi}\frac{\partial}{\partial \theta}\right]\times\left(\frac{\phi_{\mathrm{ind}}^{(0)}}{T_e} + \left.\frac{1}{f_{0e}}\frac{\partial f_{0e}}{\partial \psi_0}\right|_{v_\perp}\delta\psi\right).$$

APPENDIX E: NORMALIZED FIELD EQUATIONS

Inductive potential

$$\phi_{\mathrm{ind}} = \phi_{\mathrm{eff}} - \delta\phi. \tag{E1}$$

Evolution equation for the vector potential (12)

$$\frac{\partial \delta A_\parallel}{\partial t} = \mathbf{b}_0 \cdot \nabla \phi_{\mathrm{ind}}. \tag{E2}$$

Perturbed magnetic field label evolution (18)

$$\frac{\partial \delta\psi}{\partial t} = -\frac{\partial \phi_{\mathrm{ind}}}{\partial \alpha_0}. \tag{E3}$$

Adiabatic response equation (16)

$$\frac{\phi_{\mathrm{eff}}^{(0)}}{T_e} = \frac{\delta n_e}{n_0} - \frac{\delta\psi}{n_0}\frac{\partial n_0}{\partial \psi_0}. \tag{E4}$$

Gyrokinetic Poisson equation (5)

$$\frac{Z_i^2 n_i}{T_i}(\phi - \tilde{\phi}) = Z_i \delta n_i - \delta n_e. \tag{E5}$$

Poisson equation for the zonal component (43)

$$\frac{n_i m_i}{B_0^2}\nabla_\perp^2\langle\phi\rangle = -(1 - \rho_i^2\nabla_\perp^2)(Z_i\langle\delta n_i\rangle - \langle\delta n_e\rangle), \tag{E6}$$

where $\rho_i^2 = m_i T_i/(Z_i^2 B_0^2)$.

Zonal component of vector potential (42)

$$\langle A_\parallel\rangle = \frac{m_e}{m_p}(Z_i\langle\delta u_{\parallel i}\rangle - \langle\delta u_{\parallel e}^{(1)}\rangle). \tag{E7}$$

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