Nonlocal Theory of Energetic-Particle-Induced Geodesic Acoustic Mode

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Previous work

- EP induced GAM-like mode observed [H. Berk, NF; R. Nazikian, PRL]
  - \( n=0 \) \( m=1 \) density perturbation
  - frequency smaller than GAM frequency
- Theory on EGAM [G. Fu, PRL]
  - fluid model for thermal plasmas, kinetic model for EPs
  - threshold condition on EPs density due to local properties \( \omega_g/\omega_b \)
  - nonlocal property due to nonuniform EP profile
- Continuum of GAM on EGAM excitation [F. Zonca, IAEA]
  - \( L_b \gg L_{GAM} \Rightarrow \) fishbone like dispersion relation, threshold due to continuum damping
  - localized EP + \( L_b \ll L_{GAM} \Rightarrow \) bound state, exponentially small radiative damping \( \Rightarrow \) present work
System studied, Mode equation

- GAM continuous spectrum, circulating EPs, small drift orbit limit, drive from resonance with EPs transit frequency
- Surface averaged quasi-neutrality condition with adiabatic electrons
  \[
  \delta n_i = \delta n_{ic} + \delta n_{ih} = 0 \quad (1)
  \]
- Thermal ions perturbed density in small drift orbit limit
  \[
  \delta n_{ic} = \frac{e}{m} n_0 k_r^2 \frac{1}{\Omega_i^2} \left( -1 + \frac{\omega_g(r)}{\omega^2} \right) \delta \phi \quad (2)
  \]
EP response

- Perturbed EP distribution function

\[ \delta f = e \partial_E F_{0h} \delta \phi / m + \exp[i(m_i c)/(eB^2)] k \times B \cdot v \] \delta H_g \]

- Gyrokinetic equation

\[ (\omega - \omega_d + i \omega_t \frac{\partial}{\partial \theta}) \delta H_g = -\frac{e}{m} \frac{\partial F_{0h}}{\partial E} J_0(k_{\perp} \rho_L) \omega \delta \phi \] \hspace{1cm} (3)

\[ \delta f_h = \frac{e}{m} \frac{\partial F_{0h}}{\partial E} \delta \phi \left( 1 - \sum_{n=-\infty}^{\infty} \frac{J_n^2(\lambda_d)}{1 - nv_{||}/qR_0 \omega} \right) \] \hspace{1cm} (4)
Single pitch angle slowing-down EP

- Single pitch angle slowing-down EP
  \[ F_{oh} = c_0(r)\delta(\Lambda - \Lambda_0)H_E, \quad H_E = 1/(E^{3/2} + E_c^{3/2}) \]

- Keep up to \((k\rho_{dh})^2\)
  \[ \delta n_{ih} = ACC_0(r)\omega^2 \int_{E_c}^{E_b} \frac{dE}{E[2E(1 - \Lambda_0 B) - \omega^2 q^2 R^2]} \]

- for \(\text{Im}(\delta n_h) > 0\)
  \[ C \equiv \frac{(2 - \Lambda_0 B)}{(1 - \Lambda_0 B)^{3/2}} \left[ -\frac{3}{4}(2 - \Lambda_0 B) + \frac{3\Lambda_0^2 B^2 - 4\Lambda_0 B + 4}{4(1 - \Lambda_0 B)} \right] > 0 \]

  \[ \Rightarrow \text{threshold for local EGAM excitation} \]

  \[ \Lambda_0 B > \frac{2}{5} \quad (5) \]
Local dispersion relation of EGAM

-1 + \frac{\omega_g^2(r)}{\omega^2} + \beta_h(r) \ln \left(1 - \frac{\omega_b^2}{\omega^2}\right) = 0

in which,

\begin{align*}
\beta_h(r) &= \frac{\sqrt{1 - \Lambda_0 B q^2 C} \ n_b(r)}{8 \ln (E_b/E_c) \ n_0} \text{ density ratio} \\
\omega_b &= \sqrt{2E_b(1 - \Lambda_0 B)/(qR)} \text{ transit frequency of EP beam}
\end{align*}

- \omega_b > \omega_g \Rightarrow \text{GAM-like branch, } \omega \approx \omega_g. \text{ No threshold on EP density;} \\
- \omega_b < \omega_g \Rightarrow \text{Beam-like branch, } \omega \approx \omega_b. \text{ Threshold on EP density}\ [G. Fu, PRL]; \\
- \text{Most unstable mode: } \omega_b^2 = 2\omega_g^2.
Expanding $J_1^2(\lambda_d)$ to next order:

\[
\frac{\partial}{\partial r} \left( \rho_{dh}^2 \beta_h(r) - \alpha \rho_{dc}^2 \right) \frac{\partial}{\partial r} \delta E_r \\
+ \left[ -1 + \frac{\omega_g^2(r)}{\omega^2} + \beta_h(r) \ln \left( 1 - \frac{\omega_b^2}{\omega^2} \right) \right] \delta E_r = 0
\]

- $\rho_{dh}^2 \beta_h(r)$: FOW of EP; Contain both a $\ln(1 - \omega_b^2/\omega^2)$ and a $\ln(1 - 4\omega_b^2/\omega^2)$ term $\Rightarrow$ second harmonic of EGAM (?)
- $\alpha \rho_{dc}^2$: FOW of thermal ions[Zonca, EPS], Important for EGAM nonlocal properties!
EP radial localization

- sharply radially localized EPs beam, $L_b \ll L_{GAM}$
- GAM frequency monotonically decreasing with $r$. at $r_b$,
  $\omega_b < \omega_{GAM}; \ r_c: \ \omega_b \approx \omega_{GAM}$
- further assume $\rho_{dh}^2 \beta_h(r_b) \gg \alpha \rho_{dc}^2$

\[ n_b(r)/n_b(r_b) \]
\[ \omega_g(r)/\omega_b \]
Nonlocal dispersion relation

- Quantization condition with radiative damping due to GAM continuum $\Rightarrow$ nonlocal dispersion relation

\[ e^{2iW_1} = \frac{e^{2iW_2} + 1}{e^{2iW_2} - 1} \Rightarrow W_1 = (l + \frac{1}{2})\pi - ie^{2iW_2} \]

- Real frequency of nonlocal EGAM

\[ W_{1r}(\omega_r) = 0 \]
Threshold condition due to radiative damping

- Growth rate at marginal instability

\[
\gamma = -\frac{W_1}{\partial W_1 / \partial \omega} - \frac{e^{2iW_2}}{\partial W_1 / \partial \omega}
\]

- Drive from wave-EP resonance, radiative damping due to GAM continuum

- Threshold on EP density due to radiative damping

\[L_1 = \infty\] uniform GAM frequency profile, no threshold;

\[L_3 < L_2 < L_1\] threshold!
Mode structure

- $\delta E_r$: trapped at where EP drive is strongest, with a small radiative
- $\delta E_i$: propagation
Conclusion

- Derived the local dispersion relation of EGAM driven by wave-EP resonance in the small drift orbit limit;
- For single pitch angle slowing-down EPs, derived the threshold on pitch angle;
- FOW of both EP and bulk ions. nonlocal dispersion relation and mode structure of EGAM. Threshold due to radiative damping.
Importance of the parallel nonlinearity in the self-interaction of the geodesic acoustic mode

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Abstract. Gyrokinetic theory and simulation find that the nonlinear self-interactions of the long wavelength geodesic acoustic mode (GAM) in toroidal plasmas cannot efficiently generate the second harmonic due to a cancellation between the perpendicular convective nonlinearity and the parallel nonlinearity, which is neglected in most of gyrokinetic theory and simulation. Other mechanisms beyond conventional GAM theory are required to explain recent experimental observations of the excitation of the GAM second harmonic.
Figure 1. GTC simulations without parallel nonlinearity: Panels (a) and (b) are the time evolution and frequency spectrum of $E_r$ in a small amplitude simulation. In panel (a) the symbols are simulation data, the solid line is a numerical fitting. Panels (c) and (d) are frequency spectrum of $E_r$ and $\delta n_0$, respectively, in a large amplitude simulation.
Figure 2. The second harmonic generation rate (a) and the primary GAM damping rate (b) versus amplitude of the primary GAM.
Figure 3. Density mode structures of the zero-frequency harmonic (top panels), the primary harmonic (middle panels) and the second harmonic (bottom panels). Left column is real components and right column is imaginary components.
Figure 4. Dependence of the poloidal angle shift $\Delta \theta$ on the inverse aspect ratio $\epsilon$. These simulations are done with large torus and small simulation region $\Delta r = 0.03a$. 
Figure 5. Frequency spectrum of $\delta n_0$ (a) and time evolution of $E_r$ (b). In panel (b), the dotted line is the small amplitude simulation results, the dashed line and solid line are the large amplitude simulation with and without parallel nonlinearity results, respectively.
Figure 6. Density mode structures of the zero-frequency harmonic (top panel) and the second harmonic (bottom panels). Left column is real components and right column is imaginary component.
3. Nonlinear Gyrokinetic Theory

In order to better understand the GAM nonlinear interactions, we use nonlinear gyrokinetic theory to compare with the GTC simulations of the generation of the second harmonic. We consider a large aspect ratio axisymmetric tokamak with equilibrium magnetic field given by

$$B = B_0 (e^\xi/(1 + e \cos \theta) + (e/q)e_\theta),$$

where $\xi$ and $\theta$ are respectively the toroidal and poloidal angles of the torus.

We use the nonlinear gyrokinetic equation, in its phase space-conserving form[35],

$$\frac{\partial}{\partial t} (B^*_\parallel F) + \frac{\partial}{\partial X} (B^*_\parallel \dot{X} F) + \frac{\partial}{\partial W} (B^*_\parallel \dot{W} F) = 0,$$

where $F = F(Z, t)$, $Z = (X, W, \mu)$ is the five-dimensional gyrocenter phase space with $X$, $W$ and $\mu$ being the gyrocenter position, the parallel velocity, and the magnetic moment, respectively. $B^*_\parallel = \hat{b} \cdot B^*$, $\hat{b} = B/B$, $B^* = B + (B/\Omega) \nabla \times (W\hat{b})$ and $\nabla \equiv \partial/\partial X$. We have, in Eq.(6), further more, that

$$\dot{X} = \frac{B}{\Omega B^*_\parallel} \times \nabla H + \frac{B^*}{B^*_\parallel} \frac{\partial}{\partial W} H,$$

$$\dot{W} = - \frac{B^*_\parallel}{B^*} \cdot \nabla H,$$

$$H = \mu B + W^2/2 + (e/m) \langle \delta \phi \rangle \equiv H_0 + \delta H.$$

Here $\delta \phi$ is the perturbed scalar potential, $\Omega$ is the gyrofrequency, $\langle \cdots \rangle$ denotes average over the gyrophase angle while holding $Z$ constant. Here, for weakly nonlinear calculations, we keep only the linear perturbations $\delta H$ in the Hamiltonian $H$. Here we note the gyrocenter phase-space conservation property:

$$0 = \nabla \cdot (B^*_\parallel \dot{X}) + \frac{\partial}{\partial W} (B^*_\parallel \dot{W}),$$

which is exact to all orders in the gyrocenter analysis.

The nonlinear gyrokinetic equation Eq.(6), in the case of the second harmonic generation, can be rewritten as

$$\left( \frac{\partial}{\partial t} + i \omega_d \right) \delta F'^I + \delta X'^I \cdot \frac{\partial}{\partial X} \delta F'^I + \delta \dot{W}'^I \cdot \frac{\partial}{\partial W} \delta F'^I = 0,$$

in which, $\omega_d = \omega_d \sin \theta = -k_r m_i (v^2_\perp + v^2_\parallel) \sin \theta/(eBR)$ is the magnetic drift associated with the geodesic curvature, the superscripts $I$ and $II$ represent the primary and the second harmonic, respectively. Here, we consider only the lowest order nonlinear effects, thus we ignore the $O(\omega_d/\omega) \approx O(k_r \rho_i)$ term. For the electrostatic case, we have

$$\delta \dot{X}'^I = \frac{1}{B} \hat{b} \times \nabla \langle \delta \phi'^I \rangle,$$

$$\delta \dot{W}'^I = - \frac{e}{m} \hat{b} \cdot \nabla \langle \delta \phi'^I \rangle - \frac{1}{B} \nabla \times (W\hat{b}) \cdot \nabla \langle \delta \phi'^I \rangle.$$

We note that in Eqn.(11), the second term is the usual perpendicular nonlinear convective term, and the last term is the parallel nonlinear term. In the usual gyrokinetic ordering, the
parallel nonlinear term is neglected comparing with the perpendicular nonlinear convective term:
\[
\frac{\hat{T}_b \cdot \nabla \langle \delta \phi \rangle}{\nabla \frac{\partial F}{\partial W}} \approx \frac{1}{k_r \rho_i k_\theta R} \ll 1.
\] (14)

From the linear calculation\cite{14}, we have the linear responses of ions, to the lowest order given by:
\[
\delta F = - \frac{e}{T_i} F_M \langle \delta \phi \rangle + \delta G \approx \hat{\omega}_d \frac{e}{\omega} F_M \delta \phi_{00} \sin \theta,
\] (15)
where, subscript 00 denotes \( n = 0, m = 0 \) component. The perpendicular nonlinear convective term is given by:
\[
\delta F_{nl,R}^{II} = - \frac{e^2 k_r^I}{m \Omega T_i} \hat{\omega}_d \frac{\omega}{\omega^{II}} F_0 \cos \theta (\delta \phi_{00}^I)^2.
\] (16)

The electrons can be described by the adiabatic response. If we neglect the parallel nonlinearity term, the flux surface averaged quasineutrality condition for the second harmonic yields:
\[
\left(1 - \frac{\omega^2_{GAM}(r)}{(\omega^{II})^2}\right) \delta \phi_{00}^{II} + \frac{e}{m \omega^I \omega^{II} R^2} \left(\frac{k_r^I}{k_r^{II}}\right)^2 (\delta \phi_{00}^I)^2 = 0
\] (17)
Here, the first term is obtained from the linear GAM dispersion relation, while the second term is from the contribution of the perpendicular nonlinear convective term\( \delta F_{nl,R}^{II} \). Frequency and wavenumber matching conditions yield \( \omega^{II} = 2 \omega^I, k_r^{II} = 2 k_r^I \), and \( m = n = 0 \) for the second harmonic as observed in GTC simulation. We note that, the contribution of the flux surface averaged \( \delta F_{nl,R}^{II} \) in quasi-neutrality condition comes from the toroidal coupling. The ratio between the amplitude of the second and primary harmonic of GAM is thus, given by:
\[
\left| \frac{\delta E_r^{II}}{\delta E_r^I} \right| = \frac{1}{6 S T_i} \delta \phi_{00}^I \quad \text{where} \quad S = \left( \frac{R \omega_{GAM}}{v_i} \right)
\] (18)
In GTC simulations, this formula can be normalized as:
\[
\left| \frac{\delta E_r^{II}}{\delta E_r^I} \right| = A \frac{\delta E_r^I}{B v_i}; \quad \text{where} \quad A = \frac{1}{6 S k_r^I \rho_i}
\] (19)
In our simulations with \( k_r^I \rho_i = 0.1 \) and \( S = 2.9 \), we get \( A = 0.58 \) as mentioned in Section 2.1. This indicates that the second harmonic generation rate is proportional to the intensity of the primary harmonic, which is consistent with GTC simulation results. We emphasize that there is no threshold for the second harmonic generation, since this corresponds to a driven excitation rather than a spontaneous excitation.

In the \( k_r \rho_i \ll 1 \) limit, the corresponding density perturbation is dominated by the \( m = 1 \) poloidal component:
\[
\delta n_0 = (a \cos \theta + b \sin \theta) \frac{e}{T_i} (\delta \phi_{00}^I)^2 = \sin(\theta + \Delta \theta) \frac{e}{T_i} (\delta \phi_{00}^I)^2.
\] (20)
Here, \( a = -k_r \omega_{dr}/(2r B \omega^2) \) and \( b = \epsilon \omega_{dr}(1 + \tau)/(3 S \omega T_i) \) enter, respectively, via \( \delta F_{nl,R}^{II} \) and \( \delta F_{nl}^{II} \). Meanwhile, \( \Delta \theta \) is the poloidal angle shift of the second harmonic from the primary harmonic (Figure 4), and is given by

\[
\Delta \theta = \tan^{-1}(a/b) = \tan^{-1}\left(\frac{3 \sqrt{S} k_r \rho_i}{4 (1 + \tau) \epsilon}\right);
\]

which has a dependence on both \( \epsilon \) and \( k_r \rho_i \). Figure 4 shows that GTC simulation results agree well with that predicted by Equation (21). We note that Equation (21) is valid in the large aspect ratio limit, i.e. \( \epsilon \ll 1 \).

As we point out previously, the contribution of flux surface averaged perpendicular nonlinear convective term comes from the toroidal coupling, which is smaller than the optimal ordering of perpendicular nonlinear convective term by a factor of \( r/R \). Thus, it is comparable with the parallel nonlinear term. So here we need to include the contribution of the parallel nonlinear term, which is given by

\[
\delta F_{nl,P}^{II} = \frac{1}{i \omega^{II}} \delta \hat{W}^{II} \frac{\partial}{\partial \hat{W}} (B \delta F^{I}^{I} ),
\]

where

\[
\delta F_{nl,P}^{II} = \frac{1}{i \omega^{II}} \left[ \frac{\partial}{\partial \hat{W}} (\delta F^{I}^{I} \delta \hat{W}^{I}) + \frac{\delta F^{I}^{I}}{B} \nabla \langle \delta \phi^{I} \rangle \cdot \nabla \times \hat{b} \right].
\]

The first term is a full derivative, and will vanish in the velocity space integration. The second term, gives

\[
\delta F_{nl,P}^{II} = \frac{e^2 k_r^{I} \omega_{dr}^{I} \omega_{dr}^{II}}{m \Omega T_i R \omega_{dr}^{II} \omega_{dr}^{II}} F_0 \sin^2 \theta (\delta \phi_{00}^{I})^2.
\]

This term will cancel exactly the perpendicular nonlinear convective term after surface average. So the nonlinear harmonic generation of GAM, is higher order in \( O(\omega_{dr}/\omega) \approx O(k_r \rho_i) \) effect than the parallel nonlinearity term, and is thus, ignorable. This result can be seen directly from our simulation with the parallel nonlinearity in Section 2.2, which shows that the second harmonic is suppressed by the parallel nonlinearity.

### 4. Conclusion and Discussion

Gyrokinetic theory and simulation find that nonlinear self-interactions of the geodesic acoustic mode (GAM) in toroidal plasmas cannot efficiently generate the second harmonic due to a cancellation between the perpendicular convective nonlinearity and the parallel nonlinearity for the long wavelength GAM. Other mechanisms are required to explain recent experimental observations of the excitation of the GAM second harmonic. Our finding also raises an issue of the validity of the nonlinear GAM theory proposed to explain the generation of the GAM second harmonic as observed in tokamak experiments. Our finding indicates that the toroidal geometry, and in some situations the parallel nonlinearity, are important for the correct description of the nonlinear behaviors of the fusion plasmas.